A Syntactic Realization Theorem for Justification Logics

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Outline

1. What Is Justification Logic?
2. The Problem:
   - No Realization Theorem for Some Modal Logics
   - No Uniform Realization Method
3. Choosing the Justification Axioms
4. Reminder: Cut-free Systems Are Really Essential
5. Solution: Nested Sequent Calculus
6. Realization with Fewer Operations

What Is Justification Logic?

Modal Logic

$\Box A$
"*A* is known"
"*A* is provable"

Justification Logic

$t : A$
"*A* is known for reason *t*"
"*t* is a proof of *A*"

$$t ::= c_i \mid x_i \mid t \cdot t \mid t + t \mid\, !\, t \mid ?\, t$$

$$A ::= P_i \mid \bot \mid A \to A \mid t : A$$

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What Is Justification Logic?

Modal Logic K**:**

$\Box(A \to B) \to \Box A \to \Box B$

$$\frac{A}{\Box A}$$

Justification Logic J**:**

$s : (A \to B) \to t : A \to (s \cdot t) : B$

$$\frac{A \text{ is an axiom instance}}{c_{i_n} : \cdots : c_{i_2} : c_{i_1} : A}$$

$s : A \to (s + t) : A$
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Realization Theorem

Definition (Forgetful Projection)

$$P^\circ := P \qquad \bot^\circ := \bot \qquad (A \to B)^\circ := A^\circ \to B^\circ$$
$$(t : A)^\circ := \Box A^\circ$$

$$S^\circ := \{A^\circ \mid A \in S\}$$

Realization Theorem (Artemov, Brezhnev)

$\mathsf{J}^\circ = \mathsf{K}$.

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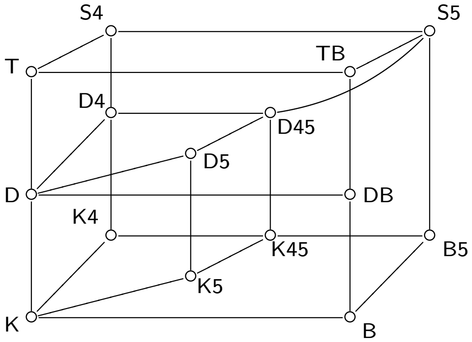

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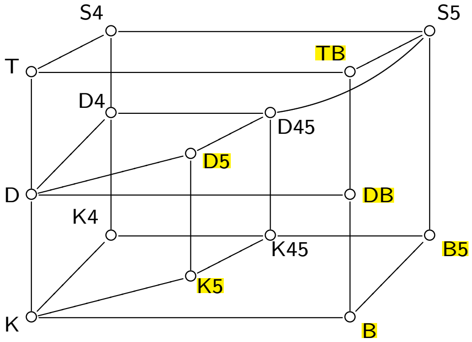

1. logics without realization

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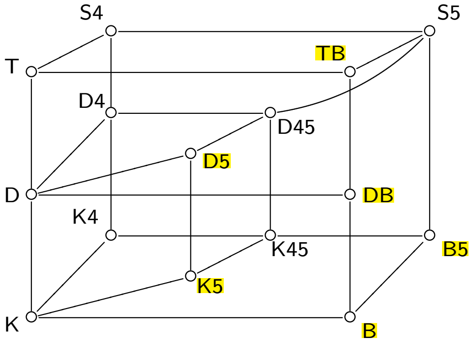

1. logics without realization
2. no uniform method

Choosing the Justification Axioms

$\Box A \to A$	$t : A \to A$	(jt)
$\Box A \to \Box\Box A$	$t : A \to\, !\, t : t : A$	(j4)
$\neg\Box A \to \Box\neg\Box A$	$\neg t : A \to\, ?\, t : (\neg t : A)$	(j5)
$\Box\bot \to \bot$	$t : \bot \to \bot$	(jd)

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$\Box\bot \to \bot$	$t : \bot \to \bot$	(jd)
	new:	
$A \to \Box\neg\Box\neg A$	$A \to\ ?t : (\neg t : \neg A)$	(jb)

Reminder: Cut-free Systems Are Really Essential

Want: if $\mathsf{ML} \vdash A$, then $\exists r\ \mathsf{JL} \vdash A^r$

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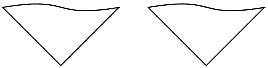

$$\frac{A \qquad A \to B}{B}$$

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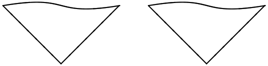
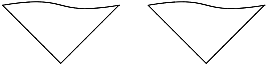


$$\frac{A \qquad A \to B}{B}$$

$\rightsquigarrow$



$$\frac{A^{r_1} \qquad A^{r_2} \to B^{r_2}}{}\ ?$$

Problem: Modus Ponens not applicable

Nested Sequents

Definition (Nested Sequent)

- $\varnothing$ is a nested sequent.
- If Σ is a nested sequent, then so is Σ, A.
- If Σ and Δ are nested sequents, then so is $\Sigma, [\Delta]$.

Definition (Corresponding Formula)

- $\underline{\varnothing} := \bot$
- $\underline{\Sigma, A} := \underline{\Sigma} \lor A$
- $\underline{\Sigma, [\Delta]} := \underline{\Sigma} \lor \Box \underline{\Delta}$

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Nested Sequents

Definition (Context)

A *context* is a nested sequent with exactly one occurrence of the symbol $\{\ \}$, not inside formulas.

Example:

$\Gamma\{\ \} = [A, [B], \{\ \}]$

$\Delta = C, [D]$

$\Gamma\{\Delta\} = [A, [B], C, [D]]$

Nested Sequents

System for Modal Logic K**:**

$$\frac{}{\Gamma\{P_i, \neg P_i\}} \qquad \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}}\ \vee \qquad \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}}\ \wedge$$

$$\frac{\Gamma\{A, A\}}{\Gamma\{A\}}\ \mathsf{ctr} \qquad \frac{\Gamma\{\Delta, \Sigma\}}{\Gamma\{\Sigma, \Delta\}}\ \mathsf{exch} \qquad \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}}\ \Box \qquad \frac{\Gamma\{[A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}\ \mathsf{k}$$

Example:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[P, \neg P, Q] \qquad \dfrac{[Q, \neg Q, \neg P]}{[\neg Q, \neg P, Q]}\,\text{exch}}{[P \wedge \neg Q, \neg P, Q]}\,\wedge}{\Diamond(P \wedge \neg Q), [\neg P, Q]}\,\text{k}}{\Diamond(P \wedge \neg Q), \Diamond \neg P, [Q]}\,\text{k}}{\Diamond(P \wedge \neg Q), \Diamond \neg P, \Box Q}\,\Box}{\Diamond(P \wedge \neg Q) \vee (\Diamond \neg P \vee \Box Q)}\,\vee}{\Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q)}\,=
$$

Rules for Extensions of K**:**

$$\frac{\Gamma\{[A]\}}{\Gamma\{\Diamond A\}}\ \mathsf{d} \qquad \frac{\Gamma\{A\}}{\Gamma\{\Diamond A\}}\ \mathsf{t} \qquad \frac{\Gamma\{[\Delta], A\}}{\Gamma\{[\Delta, \Diamond A]\}}\ \mathsf{b} \qquad \frac{\Gamma\{[\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}\ \mathsf{4}$$

$$\frac{\Gamma\{[\Delta], \Diamond A\}}{\Gamma\{[\Delta, \Diamond A]\}}\ \mathsf{5a} \qquad \frac{\Gamma\{[\Delta], [\Pi, \Diamond A]\}}{\Gamma\{[\Delta, \Diamond A], [\Pi]\}}\ \mathsf{5b} \qquad \frac{\Gamma\{[\Delta, [\Pi, \Diamond A]]\}}{\Gamma\{[\Delta, \Diamond A, [\Pi]]\}}\ \mathsf{5c}$$

Uniform Realization Theorem

Theorem

For any justification logic JL *and its corresponding modal logic* ML *we have* $\mathsf{JL}^\circ = \mathsf{ML}$.

Towards a proof

$\mathsf{JL}^\circ \subseteq \mathsf{ML}$: trivial.

$\mathsf{JL}^\circ \supseteq \mathsf{ML}$: by induction on the depth of a nested sequent proof. But there are difficulties…

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Proof Sketch

Difficulty I: Conjunction rule

$$\frac{C, A \qquad C, B}{C, A \wedge B} \wedge \quad \rightsquigarrow \quad \frac{C^{r_1} \vee A^{r_1} \qquad C^{r_2} \vee B^{r_2}}{C^{r} \vee (A^{r_1}\sigma \wedge A^{r_2}\sigma)} \; ?$$

Solution: Result by Fitting (Realization Merging):

For all C, r_1, r_2 $(*)$ there are r, σ s.t.

$\vdash C^{r_1}\sigma \rightarrow C^{r}$ and

$\vdash C^{r_2}\sigma \rightarrow C^{r}$.

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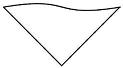

$$\dfrac{\neg\Box\neg P,\ \neg\Box\neg P}{\neg\Box\neg P}\ \text{ctr}$$

Proof Sketch

Difficulty II: Contraction rule

$$\frac{\neg\Box\neg P,\ \neg\Box\neg P}{\neg\Box\neg P}\,\text{ctr} \qquad \rightsquigarrow \qquad \frac{\neg y : \neg P \ \vee\ \neg z : \neg P}{}\ ?$$

Problem:

$$\nvdash\ \neg y : \neg P \ \vee\ \neg z : \neg P \ \to\ \neg x : \neg P$$

for any variable x.

Proof Sketch

Difficulty II: Contraction rule

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Problem:

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for any variable x.

Solution: we have

$$\vdash\ \big(\neg y : \neg P \ \vee\ \neg z : \neg P\big)\sigma \ \rightarrow\ \neg x : \neg P$$

for $\sigma = \{y \mapsto x, z \mapsto x\}$.

Realization with Fewer Operations

Theorem (Pacuit, Rubtsova)

$$\mathsf{JT45}^\circ = \mathsf{S5}$$

Theorem

$$\mathsf{JT5}^\circ = \mathsf{S5}$$
$$\mathsf{JB45}^\circ = \mathsf{JB5}^\circ = \mathsf{KB5}$$

Conclusion

Results:

- Realization Theorem for new modal logics
- New realizations with fewer operations
- Constructive, uniform proof for 15 normal modal logics

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Outlook:

- Realizations for all 32 *axiomatizations* of our modal logics
- How to choose the operations?
- Find cut-free systems for justification logics