

# A Syntactic Realization Theorem for Justification Logics

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# Outline

1. What Is Justification Logic?
2. The Problem:
  - ▶ No Realization Theorem for Some Modal Logics
  - ▶ No Uniform Realization Method
3. Choosing the Justification Axioms
4. Reminder: Cut-free Systems Are Really Essential
5. Solution: Nested Sequent Calculus
6. Realization with Fewer Operations

# What Is Justification Logic?

## Modal Logic

$\Box A$

“A is known”

“A is provable”

## Justification Logic

$t : A$

“A is known for reason  $t$ ”

“ $t$  is a proof of  $A$ ”

$t ::= c_i \mid x_i \mid t \cdot t \mid t + t \mid !t \mid ?t$

$A ::= P_i \mid \perp \mid A \rightarrow A \mid t : A$

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# What Is Justification Logic?

## Modal Logic K:

$$\Box(A \rightarrow B) \rightarrow \Box A \rightarrow \Box B$$

$$\frac{A}{\Box A}$$

## Justification Logic J:

$$s : (A \rightarrow B) \rightarrow t : A \rightarrow (s \cdot t) : B$$

$$\frac{A \text{ is an axiom instance}}{c_{i_n} : \dots : c_{i_2} : c_{i_1} : A}$$

$$s : A \rightarrow (s + t) : A$$

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# Realization Theorem

## Definition (Forgetful Projection)

$$P^\circ := P \quad \perp^\circ := \perp \quad (A \rightarrow B)^\circ := A^\circ \rightarrow B^\circ$$

$$(t : A)^\circ := \Box A^\circ$$

$$S^\circ := \{A^\circ \mid A \in S\}$$

## Realization Theorem (Artemov, Brezhnev)

$$J^\circ = K.$$



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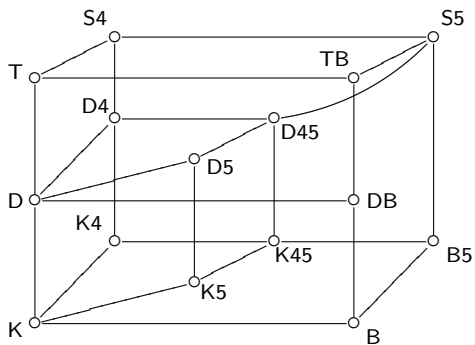
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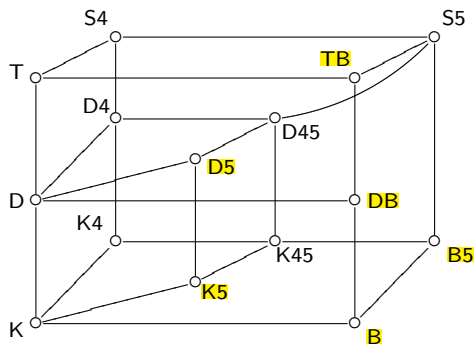
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# The Problem

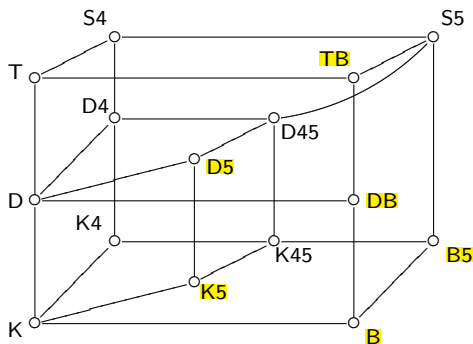


# The Problem



1. logics without realization

# The Problem



1. logics without realization
2. no uniform method

## Choosing the Justification Axioms

$$\Box A \rightarrow A \qquad t : A \rightarrow A \qquad \text{(jt)}$$

$$\Box A \rightarrow \Box \Box A \qquad t : A \rightarrow ! t : t : A \qquad \text{(j4)}$$

$$\neg \Box A \rightarrow \Box \neg \Box A \qquad \neg t : A \rightarrow ? t : (\neg t : A) \qquad \text{(j5)}$$

$$\Box \perp \rightarrow \perp \qquad t : \perp \rightarrow \perp \qquad \text{(jd)}$$

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$$\Box \perp \rightarrow \perp \qquad t : \perp \rightarrow \perp \qquad \text{(jd)}$$

new:

$$A \rightarrow \Box \neg \Box \neg A \qquad A \rightarrow ? t : (\neg t : \neg A) \qquad \text{(jb)}$$

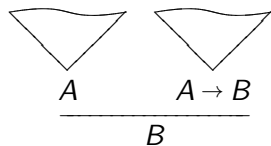
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Want: if  $ML \vdash A$ , then  $\exists r \text{ JL} \vdash A^r$

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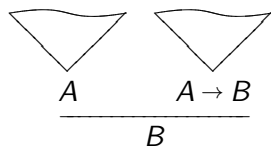




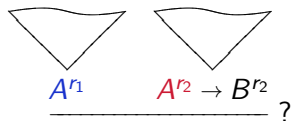
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$\rightsquigarrow$



Problem: Modus Ponens not applicable

# Nested Sequents

## Definition (Nested Sequent)

- ▶  $\emptyset$  is a nested sequent.
- ▶ If  $\Sigma$  is a nested sequent, then so is  $\Sigma, A$ .
- ▶ If  $\Sigma$  and  $\Delta$  are nested sequents, then so is  $\Sigma, [\Delta]$ .

## Definition (Corresponding Formula)

- ▶  $\underline{\emptyset} := \perp$
- ▶  $\underline{\Sigma, A} := \underline{\Sigma} \vee A$
- ▶  $\underline{\Sigma, [\Delta]} := \underline{\Sigma} \vee \Box \underline{\Delta}$

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# Nested Sequents

## Definition (Context)

A *context* is a nested sequent with exactly one occurrence of the symbol  $\{ \}$ , not inside formulas.

Example:

$$\Gamma\{ \} = [A, [B], \{ \}]$$

$$\Delta = C, [D]$$

$$\Gamma\{\Delta\} = [A, [B], C, [D]]$$

# Nested Sequents

**System for Modal Logic K:**

$$\frac{}{\Gamma\{P_i, \neg P_i\}} \quad \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}} \vee \quad \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}} \wedge$$

$$\frac{\Gamma\{A, A\}}{\Gamma\{A\}} \text{ctr} \quad \frac{\Gamma\{\Delta, \Sigma\}}{\Gamma\{\Sigma, \Delta\}} \text{exch} \quad \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \Box \quad \frac{\Gamma\{[A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}} \text{K}$$

# Nested Sequents

**Example:**

$$\frac{\frac{\frac{[P, \neg P, Q]}{[P \wedge \neg Q, \neg P, Q]} \wedge}{\diamond(P \wedge \neg Q), [\neg P, Q]} k}{\diamond(P \wedge \neg Q), \diamond \neg P, [Q]} k}{\diamond(P \wedge \neg Q), \diamond \neg P, \Box Q} \Box}{\diamond(P \wedge \neg Q) \vee (\diamond \neg P \vee \Box Q)} \vee}{\Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q)} =$$

# Nested Sequents

## Rules for Extensions of K:

$$\frac{\Gamma\{A\}}{\Gamma\{\Diamond A\}} \text{ d}$$

$$\frac{\Gamma\{A\}}{\Gamma\{\Diamond A\}} \text{ t}$$

$$\frac{\Gamma\{[\Delta], A\}}{\Gamma\{[\Delta, \Diamond A]\}} \text{ b}$$

$$\frac{\Gamma\{[\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}} \text{ 4}$$

$$\frac{\Gamma\{[\Delta], \Diamond A\}}{\Gamma\{[\Delta, \Diamond A]\}} \text{ 5a}$$

$$\frac{\Gamma\{[\Delta], [\Pi, \Diamond A]\}}{\Gamma\{[\Delta, \Diamond A], [\Pi]\}} \text{ 5b}$$

$$\frac{\Gamma\{[\Delta, [\Pi, \Diamond A]]\}}{\Gamma\{[\Delta, \Diamond A, [\Pi]]\}} \text{ 5c}$$

# Uniform Realization Theorem

## Theorem

*For any justification logic JL and its corresponding modal logic ML we have  $JL^\circ = ML$ .*

## Towards a proof

$JL^\circ \subseteq ML$ : trivial.

$JL^\circ \supseteq ML$ : by induction on the depth of a nested sequent proof.  
But there are difficulties...



# Uniform Realization Theorem

## Theorem

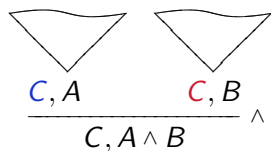
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## Difficulty I: Conjunction rule



$\rightsquigarrow$



Solution: Result by Fitting  
(Realization Merging):

For all  $C, r_1, r_2$  (\*) there are  $r, \sigma$  s.t.

$\vdash C^1\sigma \rightarrow C^r$  and

$\vdash C^2\sigma \rightarrow C^r.$



## Difficulty I: Conjunction rule

$$\frac{\begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \\ C, A \end{array} \quad \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \\ C, B \end{array}}{C, A \wedge B} \wedge$$

$\rightsquigarrow$

$$\frac{\begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \\ C^{r_1} \vee A^{r_1} \end{array} \quad \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \\ C^{r_2} \vee B^{r_2} \end{array}}{C^r \vee (A^{r_1} \sigma \wedge A^{r_2} \sigma)} ?$$


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$\vdash C^{r_1} \sigma \rightarrow C^r$  and

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## Difficulty II: Contraction rule


$$\frac{\neg \square \neg P, \neg \square \neg P}{\neg \square \neg P} \text{ctr}$$



## Difficulty II: Contraction rule

$$\frac{\begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array}}{\neg \square \neg P, \neg \square \neg P} \text{ctr} \quad \rightsquigarrow \quad \frac{\begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array}}{\neg y : \neg P \vee \neg z : \neg P} ?$$

Problem:

$$\not\vdash \neg y : \neg P \vee \neg z : \neg P \rightarrow \neg x : \neg P$$

for any variable  $x$ .

Solution: we have

$$\vdash (\neg y : \neg P \vee \neg z : \neg P)\sigma \rightarrow \neg x : \neg P$$

for  $\sigma = \{y \mapsto x, z \mapsto x\}$ .

# Realization with Fewer Operations

Theorem (Pacuit, Rubtsova)

$$JT45^\circ = S5$$

Theorem

$$JT5^\circ = S5$$

$$JB45^\circ = JB5^\circ = KB5$$



# Conclusion

## Results:

- ▶ Realization Theorem for new modal logics
- ▶ New realizations with fewer operations
- ▶ Constructive, uniform proof for 15 normal modal logics

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## Outlook:

- ▶ Realizations for all 32 *axiomatizations* of our modal logics
- ▶ How to choose the operations?
- ▶ Find cut-free systems for justification logics