Cut-elimination and Proof Search for Bi-Intuitionistic Tense Logic

Rajeev Goré, Alwen Tiu and Linda Postniece

Logic and Computation Group
School of Computer Science
The Australian National University
Canberra

August 25, 2010

Propositional Bi-Intuitionistic Logic (Rauszer)

Int: intuitionistic logic with $\wedge, \vee, \to, \bot$

$x \Vdash A \to B$ iff $\forall y \geq x. y \Vdash A \Rightarrow y \Vdash B$ (every successor)

DInt: dual intuitionistic logic with $\wedge, \vee, \prec, \top$

$x \Vdash A \prec B$ iff $\exists y \leq x. y \Vdash A \,\&\, y \nVdash B$ (some predecessor)

BiInt: Int and DInt plus axioms like $A \to (B \vee (A \prec B))$ ...

Adjunctions: $(\wedge, \to)$ and $(\vee, \prec)$

$(A \wedge B) \to C$ *iff* $B \to (A \to C)$ *iff* $A \to (B \to C)$

$A \to (B \vee C)$ *iff* $(A \prec B) \to C$ *iff* $(A \prec C) \to B$

Propositional Classical Tense Logic Kt

Kf: classical modal logic with $\wedge, \vee, \rightarrow, \perp, \Box, \Diamond$

$x \Vdash \Box A$ iff $\forall y. R(x, y) \Rightarrow y \Vdash A$ (every successor)

$x \Vdash \Diamond A$ iff $\exists y. R(x, y)$ & $y \Vdash A$ (some successor)

Kp: classical modal logic with $\wedge, \vee, \rightarrow, \perp, \blacksquare, \blacklozenge$

$x \Vdash \blacksquare A$ iff $\forall y. R(y, x) \Rightarrow y \Vdash A$ (every predecessor)

$x \Vdash \blacklozenge A$ iff $\exists y. R(y, x)$ & $y \Vdash A$ (some predecessor)

Kt: Kf plus Kp plus interactions axioms $A \rightarrow \Box\blacklozenge A$ and $A \rightarrow \blacksquare\Diamond p$

Adjunctions: $(\blacklozenge, \Box)$ and $(\Diamond, \blacksquare)$

$$\blacklozenge A \rightarrow B \quad \textit{iff} \quad A \rightarrow \Box B$$
$$\Diamond A \rightarrow B \quad \textit{iff} \quad A \rightarrow \blacksquare B$$

Propositional Bi-Intuitionistic Tense Logic

BiKt: bi-intuitionistic logic $\wedge, \vee, \rightarrow, \bot, \prec, \top$ with $\blacksquare, \blacklozenge, \Box, \Diamond$

$x \Vdash A \rightarrow B$	iff	$\forall y \geq x . y \Vdash A \Rightarrow y \Vdash B$	(every $\leq$-successor)
$x \Vdash A \prec B$	iff	$\exists y \leq x . y \Vdash A \;\&\; y \nVdash B$	(some $\leq$-predecessor)
$x \Vdash \blacklozenge A$	iff	$\exists y . R_{\Box}(y, x) \;\&\; y \Vdash A$	(some $R_{\Box}$-predecessor)
$x \Vdash \Box A$	iff	$\forall y . R_{\Box}(x, y) \Rightarrow y \Vdash A$	(every $R_{\Box}$-successor)
$x \Vdash \Diamond A$	iff	$\exists y . R_{\Diamond}(x, y) \;\&\; y \Vdash A$	(some $R_{\Diamond}$-successor)
$x \Vdash \blacksquare A$	iff	$\forall y . R_{\Diamond}(y, x) \Rightarrow y \Vdash A$	(every $R_{\Diamond}$-predecessor)

Three binary relations: no explicit connections between $R_{\Diamond}$ and $R_{\Box}$

Persistence: $\forall v \geq w.\ w \in V(p) \Rightarrow v \in V(p)$

Reverse-Persistence: $\forall v \leq w.\ w \notin V(p) \Rightarrow v \notin V(p)$

Frame Conditions:

F1$\Diamond$ if $x \leq y \;\&\; x R_{\Diamond} z$ then $\exists w.\ y R_{\Diamond} w \;\&\; z \leq w$

F2$\Box$ if $x R_{\Box} y \;\&\; y \leq z$ then $\exists w.\ x \leq w \;\&\; w R_{\Box} z$

Modular Proof Theory Suitable for Backward Proof Search

LBiKt: a display-like shallow nested sequent calculus

Cut-elimination: to transform **LBiKt**-derivations into cut-free ones

DBiKt: a nested sequent calculus using deep inference rules

Equiderivability: between cut-free **LBiKt** and (cut-free) **DBiKt**

Modularity: ability to restrict and extend base systems using rules that capture particular axioms

Classical collapse: by the addition of structural rules

Proof Search: by restricting rules further ... termination still open

Soundness: wrt the Kripke semantics

Completeness: wrt the Kripke semantics (in extended version)

What's Wrong With Gentzen's Traditional Sequents?

Modal logics: works for some well-known logics

$$\frac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A}\,\Box R \quad \frac{\Gamma, \Box A, A \vdash \Delta}{\Gamma, \Box A \vdash \Delta}\,T \quad \frac{\Gamma, \Box A, A \vdash \Delta}{\Box\Gamma, \Box A \vdash \Box\Delta}\,\Box 4R$$

Tense logics: creates formulae not in original end-sequent

$$\frac{\Gamma \vdash A}{\blacksquare\Delta, \Box\Gamma \vdash \Box A}\,\Box R?? \quad \frac{\dfrac{?}{\blacklozenge\blacksquare\Delta, \Gamma \vdash A}}{\blacksquare\Delta, \Box\Gamma \vdash \Box A}\,\Box R?$$

Need extra "machinery": that extends Gentzen's comma

Existing Proof-theoretic Methodologies

Display Calculi: extremely modular but bad for proof search

Nested Sequents: modular but display-like hence bad for proof search

Labelled Sequents: explicitly represent Kripke semantics

Hyper-Sequents: no work on int tense logics to our knowledge

Our work: extension of nested sequents via deep inference

Formulae, Structures and Nested Sequents

Formulae:

$$A ::= p \mid \top \mid \bot \mid A \to A \mid A \prec A \mid A \wedge A \mid A \vee A \mid \Box A \mid \Diamond A \mid \blacksquare A \mid \blacklozenge A$$

Structures:

$$X := \emptyset \mid A \mid (X, X) \mid X \triangleright X \mid \circ X \mid \bullet X$$

Nested Sequent: is a structure of the form $X \triangleright Y$

Assume: Comma is associative and commutative with unit $\emptyset$

Intuition: think of sequents as formation trees

Related Work: generalises Kashima's nested sequents, Brünnler's deep sequents, and Poggiolesi's tree-hypersequents

Formula Translation of Nested Sequents

$$\tau(X \triangleright Y) \quad = \quad \tau^-(X) \to \tau^+(Y)$$

$$\begin{array}{lcl} \tau^-(A) & = & A \\ \tau^-(X, Y) & = & \tau^-(X) \land \tau^-(Y) \\ \tau^-(X \triangleright Y) & = & \tau^-(X) \prec \tau^+(Y) \\ \tau^-(\circ X) & = & \Diamond \tau^-(X) \\ \tau^-(\bullet X) & = & \blacklozenge \tau^-(X) \end{array} \qquad \begin{array}{lcl} \tau^+(A) & = & A \\ \tau^+(X, Y) & = & \tau^+(X) \lor \tau^+(Y) \\ \tau^+(X \triangleright Y) & = & \tau^-(X) \to \tau^+(Y) \\ \tau^+(\circ X) & = & \Box \tau^+(X) \\ \tau^+(\bullet X) & = & \blacksquare \tau^+(X) \end{array}$$

Gentzen toggles (different from display calculi)

comma: is $\land / \lor$ on left/right of sequent

$\circ$: is interpreted as $\Diamond / \Box$ on left/right of sequent

$\bullet$: is interpreted as $\blacklozenge / \blacksquare$ on left/right of sequent

$\triangleright$: is interpreted as $\prec$ / $\to$ on left/right of sequent

Some Logical Rules of the Shallow System **LBiKt**

$$\frac{}{X, A \rhd A, Y}\, id \qquad \frac{X, A \rhd B}{X \rhd A \to B, Y} \to_R \qquad \frac{X \rhd A, Y \qquad X, B \rhd Y}{X \rhd A \prec B, Y} \prec_R$$

$$\frac{A \rhd X}{\Box A \rhd \circ X} \Box_L \qquad \frac{X \rhd \bullet A}{X \rhd \blacksquare A} \blacksquare_R \qquad \frac{\circ A \rhd X}{\Diamond A \rhd X} \Diamond_L \qquad \frac{X \rhd A}{\bullet X \rhd \blacklozenge A} \blacklozenge_R$$

shallow: rules are only applicable at top level

modalities: must be displayed as whole of left/right hand side

structure: can be created or removed (backwards)

need: structural rules to bring formulae to top level

Some Structural Rules of the Shallow System **LBiKt**

$$\frac{X, Y, Y \triangleright Z}{X, Y \triangleright Z}\, c_L \quad \frac{X \triangleright Z}{X, Y \triangleright Z}\, w_L \quad \frac{\bullet X \triangleright Y}{X \triangleright \circ Y}\, rp_\circ \quad \frac{X_1, X_2 \triangleright Y_2}{X_1 \triangleright (X_2 \triangleright Y_2)}\, \triangleright_R$$

$$\frac{(X_1 \triangleright Y_1), X_2 \triangleright Y_2}{X_1, X_2 \triangleright Y_1, Y_2}\, s_L \qquad \frac{X_1 \triangleright Y_1, A \qquad A, X_2 \triangleright Y_2}{X_1, X_2 \triangleright Y_1, Y_2}\, cut$$

$rp_\circ$: allows us to move modalities aside

s_L and $\triangleright_R$: allow us to unravel bi-intuitionistic parts

cut: applied only on formulae at top level

Backward proof search: cut, weakening and contraction are bad

Need: to compile the structural rules into the logical rules

Example of Shallow Derivation

$$\frac{X, A \triangleright B}{X \triangleright A \rightarrow B, Y} \rightarrow_R \quad \frac{X \triangleright \circ A}{X \triangleright \Box A} \Box_R \quad \frac{X \triangleright A}{\bullet X \triangleright \blacklozenge A} \blacklozenge_R \quad \frac{\bullet X \triangleright Y}{X \triangleright \circ Y} rp_\circ$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \triangleright A} id}{\bullet A \triangleright \blacklozenge A} \blacklozenge_R}{A \triangleright \circ \blacklozenge A} rp_\circ}{A \triangleright \Box \blacklozenge A} \Box_R}{\emptyset \triangleright A \rightarrow \Box \blacklozenge A} \rightarrow_R$$

Shallow

$\rightarrow_r$ displays A

$\Box_r$ rewrites displayed $\Box A$

$rp_\circ$ displays $\blacklozenge A$

Cut-elimination for Shallow Inference

Theorem: If $X \triangleright Y$ is **LBiKt**-derivable then it is also **LBiKt**-derivable without using cut.

$$\dfrac{\begin{array}{c}\Psi_1 \\ X_1' \triangleright A\end{array}}{\circ(X_1') \triangleright \Diamond A}\ \Diamond_R \quad \vdots \quad X_1 \triangleright Y_1, \Diamond A \qquad (1)$$

$$\dfrac{\begin{array}{c}\Psi_2 \\ \circ A \triangleright Y_2'\end{array}}{\Diamond A \triangleright Y_2'}\ \Diamond_L \quad \vdots \quad \Diamond A, X_2 \triangleright Y_2 \qquad (2)$$

$$\circ X_1' \triangleright (X_2 \triangleright Y_2) \quad \vdots \quad \dfrac{X_1 \triangleright Y_1, (X_2 \triangleright Y_2)}{X_1, X_2 \triangleright Y_1, Y_2}\ s_R \qquad (3)$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\Psi_1 \\ X_1' \triangleright A\end{array} \quad \dfrac{\begin{array}{c}\Psi_2 \\ \circ A \triangleright Y_2'\end{array}}{A \triangleright \bullet Y_2'}\ rp_\bullet}{X_1' \triangleright \bullet Y_2'}\ cut}{\circ X_1' \triangleright Y_2'}\ rp_\bullet \quad \vdots \quad \circ X_1', X_2 \triangleright Y_2}{\circ X_1' \triangleright (X_2 \triangleright Y_2)}\ \triangleright_R \qquad (4)$$

Nested Sequent System **DBiKt** Using Deep Inference

Proof search: **LBiKt** is bad for backward proof search

Context $\Sigma[\ ]$ is a structure with a hole or a placeholder []

Filling: hole with structure X gives $\Sigma[X]$

Simple context: hole is not under the scope of $\triangleright$

Negative context: $\Sigma^{-}[]$ when hole appears to the left of the closest ancestor node labelled with $\triangleright$

Positive context: $\Sigma^{+}[]$ when hole appears to the right of the closest ancestor node labelled with $\triangleright$

Note: different from traditional notion of polarity in display calculi which is relative to the top-most "turnstile"

Beware: our deep inference is not Guglielmi's Deep Inference!

Structural (Propagation) Rules in **DBiKt**

$$\frac{\Sigma^-[A, (A, X \triangleright Y)]}{\Sigma^-[A, X \triangleright Y]} \triangleright_{L1} \qquad \frac{\Sigma^+[(X \triangleright Y, A), A]}{\Sigma^+[X \triangleright Y, A]} \triangleright_{R1} \qquad \frac{\Sigma[(X \triangleright Y, A), W \triangleright A, Z]}{\Sigma[(X \triangleright Y), W \triangleright A, Z]} \triangleright_{R2}$$

$$\frac{\Sigma^-[A, \bullet(\Box A, X)]}{\Sigma^-[\bullet(\Box A, X)]} \Box_{L1} \qquad \frac{\Sigma^+[A, \circ(\blacklozenge A, X)]}{\Sigma^+[\circ(\blacklozenge A, X)]} \blacklozenge_{R1} \qquad \frac{\Sigma[\blacksquare A, X \triangleright \bullet(A \triangleright Y), Z]}{\Sigma[\blacksquare A, X \triangleright \bullet Y, Z]} \blacksquare_{L2}$$

rules are polarity dependent : positive, negative or simple contexts

contraction: built into most rules i.e. premises contain conclusions

propagation: of formulae when read backwards

create structure: when read backwards

proof search: must tame the application of these rules

Example of Shallow and Deep Derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \triangleright A}\, id}{\bullet A \triangleright \blacklozenge A}\, \blacklozenge_R}{A \triangleright \circ \blacklozenge A}\, rp_\circ}{A \triangleright \Box \blacklozenge A}\, \Box_R}{\emptyset \triangleright A \to \Box \blacklozenge A}\, \to_R$$

Shallow

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\emptyset \triangleright (A \triangleright A, \circ(\blacklozenge A))}\, id}{\emptyset \triangleright (A \triangleright \circ(\blacklozenge A))}\, \blacklozenge_{R1}}{\emptyset \triangleright (A \triangleright \Box \blacklozenge A)}\, \Box_R}{\emptyset \triangleright A \to \Box \blacklozenge A}\, \to_R$$

Deep

$\to_r$ displays A
$\Box_r$ rewrites displayed $\Box A$
$rp_\circ$ displays $\blacklozenge A$

$\to_r$ nests A
$\Box_r$ rewrites nested $\Box A$
$\blacklozenge_{R_1}$ deeply propagates A

Equi-derivability between **LBiKt** and **DBiKt**

Right to left:

Theorem 4.1: every rule of **DBiKt** is derivable in **LBiKt**

Left to right: many technical proof-theoretic lemmas

Lemmas 4.2-4.6: **LBiKt** structural rules are **DBiKt**-admissible
(if premisses derivable then so is conclusion)

Logical: **LBiKt** rules mimicked (shallowly) in **DBiKt**
(deep rules can always be applied shallowly)

Modularity of **DBiKt** for its fragments

Purely modal nested sequent contains no occurrences of $\bullet$ nor its formula translates $\blacksquare$ and $\blacklozenge$

DInt: sub-system of **DBiKt** containing *id*, logical rules for intuitionistic connectives, and propagation rules for $\triangleright$

DIntK: is **DInt** plus the deep introduction rules for $\Box$ and $\Diamond$, and the propagation rules $\Box_{L2}$ and $\Diamond_{R2}$

DBInt: is **DInt** plus the deep introduction rules for $\prec$

Modularity: for an $\mathrm{Int}/\mathrm{BiInt}/\mathrm{IntK}$ formula A, the sequent $\emptyset \triangleright A$ is **DInt**/**DBInt**/**DIntK**-derivable iff $\emptyset \triangleright A$ is **DBiKt**-derivable

Proof: the only rules that create $\bullet$ upwards are $\blacklozenge_L$ and $\blacksquare_R$. Thus in every **DBiKt**-derivation π of an IntK formula, the internal sequents in π are purely modal, so π is also a **DIntK**-derivation

Syntactic Extensions

Ewald: $(\Diamond A \rightarrow \Box B) \rightarrow \Box(A \rightarrow B)$ $(\blacklozenge A \rightarrow \blacksquare B) \rightarrow \blacksquare(A \rightarrow B)$

$$\frac{X \triangleright \bullet Y \triangleright \bullet Z}{X \triangleright \bullet(Y \triangleright Z)} \bullet\triangleright_R \qquad \frac{X \triangleright \circ Y \triangleright \circ Z}{X \triangleright \circ(Y \triangleright Z)} \circ\triangleright_R$$

Axiomatic Extensions

$$\frac{\Sigma^-[A, \Box A]}{\Sigma^-[\Box A]} T\Box \qquad \frac{\Sigma[\Box A, X \triangleright \circ(\Box A \triangleright Y), Z]}{\Sigma[\Box A, X \triangleright \circ Y, Z]} 4\Box_L \qquad \frac{\Sigma^-[A, \circ(\Box A, X)]}{\Sigma^-[\circ(\Box A, X)]} B\Box_L$$

$$\dfrac{\dfrac{\dfrac{}{p, \Box p \triangleright p}\, id}{\Box p \triangleright p}\, T\Box}{\triangleright \Box p \rightarrow p} \rightarrow_R \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Box p \triangleright \circ(\Box p \triangleright \Box p)}\, id}{\Box p \triangleright \circ \Box p}\, 4\Box_L}{\Box p \triangleright \Box\Box p}\, \Box_R}{\triangleright \Box p \rightarrow \Box\Box p} \rightarrow_R \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{p, \circ\Box p \triangleright p}\, id}{\circ\Box p \triangleright p}\, B\Box_L}{\Diamond\Box p \triangleright p}\, \Diamond_L}{\triangleright \Diamond\Box p \rightarrow p} \rightarrow_R$$

Note: do not correspond to reflexivity, transitivity, symmetry of $R_\Box$

Classical Collapse Via Structural Rules

$$\frac{X_1, X_2 \rhd Y_1, Y_2}{(X_1 \rhd Y_1), X_2 \rhd Y_2}\, s_L^{-1} \qquad \frac{X_1, X_2 \rhd Y_1, Y_2}{X_1 \rhd Y_1, (X_2 \rhd Y_2)}\, s_R^{-1}$$

The law of the excluded middle and the law of (dual-)contradiction can then be derived as shown below:

$$\dfrac{\dfrac{\dfrac{\dfrac{p \rhd p, \bot}{(\emptyset \rhd p), p \rhd \bot}\, s_L^{-1}}{(\emptyset \rhd p) \rhd (p \to \bot)}\, \to_R}{\emptyset \rhd p, (p \to \bot)}\, s_L}{\emptyset \rhd p \vee (p \to \bot)}\, \vee_L \qquad \dfrac{\dfrac{\dfrac{\dfrac{p, \top \rhd p}{\top \rhd p, (p \rhd \emptyset)}\, s_R^{-1}}{(\top \prec p) \rhd (p \rhd \emptyset)}\, \prec_L}{p, (\top \prec p) \rhd \emptyset}\, s_R}{p \wedge (\top \prec p) \rhd \emptyset}\, \wedge_R$$

Proof Search Using the Deep System

Backward proof search strategy: proceeds in three stages: saturation, propagation and *realisation*

Saturation phase: applies (backwards) the "static rules" (i.e. those that do not create extra structural connectives) until further backward application do not lead to any progress

Propagation phase: propagates formulaes across different structural connectives

Realisation phase: applies the "dynamic rules" (i.e., those that create new structural connectives, e.g., $\rightarrow_R$)

Caveat: we do not yet have termination or completeness

Further Work

Termination: of our proof search procedure

Expressivity: what class of axiomatic extensions can we capture

Implementation: extend our previous prover for BiInt to BiKt

Rauszer's Axioms for BiInt

$$(A \to B) \to ((B \to C) \to (A \to C)) \quad (1)$$
$$A \to A \vee B \quad (2)$$
$$B \to A \vee B \quad (3)$$
$$(A \to C) \to ((B \to C) \to ((A \vee B) \to C)) \quad (4)$$
$$(A \wedge B) \to A \quad (5)$$
$$(A \wedge B) \to B \quad (6)$$
$$(C \to A) \to ((C \to B) \to (C \to (A \wedge B))) \quad (7)$$
$$(A \to (B \to C)) \to ((A \wedge B) \to C) \quad (8)$$
$$((A \wedge B) \to C) \to (A \to (B \to C)) \quad (9)$$
$$A \to (B \vee (A \prec B)) \quad (10)$$
$$(A \to B) \to (\neg B \to \neg A) \quad (11)$$
$$(A \prec B) \to \sim (A \to B) \quad (12)$$
$$((A \prec B) \prec C) \to (A \prec (B \vee C)) \quad (13)$$
$$\neg(A \prec B) \to (A \to B) \quad (14)$$
$$(A \to (B \prec B)) \to \neg A \quad (15)$$
$$\neg A \to (A \to (B \prec B)) \quad (16)$$
$$((B \to B) \prec A) \to \sim A \quad (17)$$
$$\sim A \to ((B \to B) \prec A) \quad (18)$$

MP plus From A infer $\neg \sim A$