# Cut-elimination and Proof Search for Bi-Intuitionistic Tense Logic

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## Propositional Bi-Intuitionistic Logic (Rauszer)

Int: intuitionistic logic with  $\land, \lor, \rightarrow, \bot$  $x \Vdash A \rightarrow B$  iff  $\forall y \ge x.y \Vdash A \Rightarrow y \Vdash B$  (every successor)

DInt: dual intuitionistic logic with  $\land, \lor, \neg \prec, \top$  $x \Vdash A \longrightarrow B$  iff  $\exists y \leq x.y \Vdash A \& y \nvDash B$  (some predecessor)

Bilnt: Int and DInt plus axioms like  $A \rightarrow (B \lor (A \prec B)) \dots$ 

#### Propositional Classical Tense Logic Kt

Kf: classical modal logic with  $\land, \lor, \rightarrow, \bot, \Box, \diamondsuit$  $x \Vdash \Box A$  iff  $\forall y.R(x,y) \Rightarrow y \Vdash A$ (every successor) $x \Vdash \Diamond A$  iff  $\exists y.R(x,y) \& y \Vdash A$ (some successor)

Kp: classical modal logic with  $\land, \lor, \rightarrow, \bot, \blacksquare, \blacklozenge$  $x \Vdash \blacksquare A$  iff  $\forall y. R(y, x) \Rightarrow y \Vdash A$ (every predecessor) $x \Vdash \blacklozenge A$  iff  $\exists y. R(y, x) \& y \Vdash A$ (some predecessor)

Kt: Kf plus Kp plus interactions axioms  $A \to \Box \blacklozenge A$  and  $A \to \blacksquare \diamondsuit p$ Adjunctions:  $(\diamondsuit, \Box)$  and  $(\diamondsuit, \blacksquare)$ 

$$A \to B \quad iff \quad A \to \Box B \\ \Diamond A \to B \quad iff \quad A \to \blacksquare B$$

### Propositional Bi-Intuitionistic Tense Logic

BiKt: bi-intuitionistic logic  $\land, \lor, \rightarrow, \bot, \neg \prec, \top$  with  $\blacksquare, \blacklozenge, \Box, \diamondsuit$ 

$x \Vdash A \to B$	iff	$\forall y \ge x.y \Vdash A \Rightarrow y \Vdash B$	(every $\leq$ -successor)
$x \Vdash A \longrightarrow B$	iff	$\exists y \leq x.y \Vdash A \& y \not\Vdash B$	(some $\leq$ -predecessor)
$x \Vdash \blacklozenge A$	iff	$\exists y. R_{\Box}(y, x) \& y \Vdash A$	(some $R_{\Box}$ -predecessor)
$x \Vdash \Box A$	iff	$\forall y. R_{\Box}(x, y) \Rightarrow y \Vdash A$	(every $R_{\Box}$ -successor)
$x \Vdash \Diamond A$	iff	$\exists y. R_{\Diamond}(x, y) \& y \Vdash A$	(some $R_{\Diamond}$ -successor)
$x \Vdash \blacksquare A$	iff	$\forall y. R_{\Diamond}(y, x) \Rightarrow y \Vdash A$	(every $R_{\Diamond}$ -predecessor)

Three binary relations: no explicit connections between  $R_{\Diamond}$  and  $R_{\Box}$ Persistence:  $\forall v \ge w. \ w \in V(p) \Rightarrow v \in V(p)$ Reverse-Persistence:  $\forall v \le w. \ w \notin V(p) \Rightarrow v \notin V(p)$ Frame Conditions:

F1
$$\diamond$$
 if  $x \le y \& xR_{\diamond}z$  then  $\exists w. yR_{\diamond}w \& z \le w$   
F2 $\Box$  if  $xR_{\Box}y \& y \le z$  then  $\exists w. x \le w \& wR_{\Box}z$ 

## Modular Proof Theory Suitable for Backward Proof Search

LBiKt: a display-like shallow nested sequent calculus Cut-elimination: to transform LBiKt-derivations into cut-free ones DBiKt: a nested sequent calculus using deep inference rules Equiderivability: between cut-free LBiKt and (cut-free) DBiKt

Modularity: ability to restrict and extend base systems using rules that capture particular axioms

Classical collapse: by the addition of structural rules

Proof Search: by restricting rules further ... termination still open

Soundness: wrt the Kripke semantics Completeness: wrt the Kripke semantics (in extended version) What's Wrong With Gentzen's Traditional Sequents?

Modal logics: works for some well-known logics

$$\frac{\Gamma \vdash A}{\Box \Gamma \vdash \Box A} \Box \mathsf{R} \quad \frac{\Gamma, \Box A, A \vdash \Delta}{\Gamma, \Box A \vdash \Delta} \mathsf{T} \quad \frac{\Gamma, \Box A, A \vdash \Delta}{\Box \Gamma, \Box A \vdash \Box \Delta} \Box \mathsf{4} \mathsf{R}$$

Tense logics: creates formulae not in original end-sequent

$$\frac{\Gamma \vdash A}{\blacksquare \Delta, \Box \Gamma \vdash \Box A} \Box R?? \quad \frac{?}{\blacksquare \Delta, \Gamma \vdash A} \Box R?$$

Need extra "machinery": that extends Gentzen's comma

Display Calculi: extremely modular but bad for proof search Nested Sequents: modular but display-like hence bad for proof search

Labelled Sequents: explicitly represent Kripke semantics Hyper-Sequents: no work on int tense logics to our knowledge Our work: extension of nested sequents via deep inference Formulae, Structures and Nested Sequents

#### Formulae:

$$A ::= p \mid \top \mid \bot \mid A \to A \mid A - < A \mid A \land A \mid A \lor A \mid \Box A \mid \Diamond A \mid \blacksquare A \mid \blacklozenge A \mid \blacklozenge A \mid \blacksquare A \mid \blacklozenge A$$

Structures:

$$X := \emptyset \mid A \mid (X, X) \mid X \triangleright X \mid \circ X \mid \bullet X$$

Nested Sequent: is a structure of the form  $X \triangleright Y$ 

Assume: Comma is associative and commutative with unit  $\emptyset$ 

Intuition: think of sequents as formation trees

Related Work: generalises Kashima's nested sequents, Brünnler's deep sequents, and Poggiolesi's tree-hypersequents

#### Formula Translation of Nested Sequents

$$\tau(X \triangleright Y) = \tau^{-}(X) \rightarrow \tau^{+}(Y)$$

$$\tau^{-}(A) = A \qquad \tau^{+}(A) = A$$

$$\tau^{-}(X,Y) = \tau^{-}(X) \wedge \tau^{-}(Y) \qquad \tau^{+}(X,Y) = \tau^{+}(X) \vee \tau^{+}(Y)$$

$$\tau^{-}(X \triangleright Y) = \tau^{-}(X) \longrightarrow \tau^{+}(Y) \qquad \tau^{+}(X \triangleright Y) = \tau^{-}(X) \rightarrow \tau^{+}(Y)$$

$$\tau^{-}(\circ X) = \Diamond \tau^{-}(X) \qquad \tau^{+}(\circ X) = \Box \tau^{+}(X)$$

$$\tau^{-}(\bullet X) = \blacklozenge \tau^{-}(X) \qquad \tau^{+}(\bullet X) = \Box \tau^{+}(X)$$

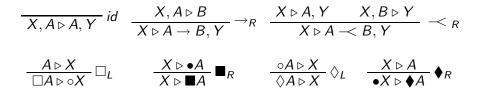
Gentzen toggles

(different from display calculi)

comma: is  $\wedge/\vee$  on left/right of sequent

- o: is interpreted as  $\Diamond/\Box$  on left/right of sequent
- •: is interpreted as  $\blacklozenge/\blacksquare$  on left/right of sequent
- $\triangleright:$  is interpreted as  $\ -\!\!\!</ \rightarrow$  on left/right of sequent

## Some Logical Rules of the Shallow System LBiKt



shallow: rules are only applicable at top level
modalities: must be displayed as whole of left/right hand side
structure: can be created or removed (backwards)
need: structural rules to bring formulae to top level

## Some Structural Rules of the Shallow System LBiKt

$$\frac{X, Y, Y \triangleright Z}{X, Y \triangleright Z} c_{L} \quad \frac{X \triangleright Z}{X, Y \triangleright Z} w_{L} \quad \frac{\bullet X \triangleright Y}{X \triangleright \circ Y} rp_{\circ} \quad \frac{X_{1}, X_{2} \triangleright Y_{2}}{X_{1} \triangleright (X_{2} \triangleright Y_{2})} \triangleright_{R}$$

$$\frac{(X_{1} \triangleright Y_{1}), X_{2} \triangleright Y_{2}}{X_{1}, X_{2} \triangleright Y_{1}, Y_{2}} s_{L} \qquad \qquad \frac{X_{1} \triangleright Y_{1}, A \quad A, X_{2} \triangleright Y_{2}}{X_{1}, X_{2} \triangleright Y_{1}, Y_{2}} cut$$

 $rp_{\circ}$ : allows us to move modalities aside  $s_L$  and  $\triangleright_R$ : allow us to unravel bi-intuitionistic parts cut: applied only on formulae at top level Backward proof search: cut, weakening and contraction are bad Need: to compile the structural rules into the logical rules Example of Shallow Derivation

$$\frac{X, A \triangleright B}{X \triangleright A \to B, Y} \rightarrow_{R} \frac{X \triangleright \circ A}{X \triangleright \Box A} \Box_{R} \frac{X \triangleright A}{\bullet X \triangleright \diamond A} \blacklozenge_{R} \frac{\bullet X \triangleright Y}{X \triangleright \circ Y} rp_{\circ}$$

$$\frac{\overline{A \triangleright A}}{\bullet A \triangleright \diamond A} \stackrel{id}{\bullet_{R}} \frac{\bullet A \triangleright \diamond A}{A \triangleright \odot \diamond A} \overline{\Box_{R}}$$

$$\frac{A \triangleright \Box \diamond A}{A \triangleright \Box \diamond A} \overline{\Box_{R}}$$

$$\emptyset \triangleright A \to \Box \diamond A \xrightarrow{\rightarrow R}$$
Shallow

 $\begin{array}{c} \longrightarrow_r \text{displays } A \\ \square_r \text{ rewrites displayed } \square A \\ rp_\circ \text{ displays } \blacklozenge A \end{array}$ 

### Cut-elimination for Shallow Inference

Theorem: If  $X \triangleright Y$  is **LBiKt**-derivable then it is also **LBiKt**-derivable without using cut.

110

# Nested Sequent System **DBiKt** Using Deep Inference

Proof search: LBiKt is bad for backward proof search
Context Σ[] is a structure with a hole or a placeholder []
Filling: hole with structure X gives Σ[X]
Simple context: hole is not under the scope of ▷
Negative context: Σ<sup>-</sup>[] when hole appears to the left of the closest ancestor node labelled with ▷
Pacifies context: Σ<sup>+</sup>[] when hole appears to the right of the

Positive context:  $\Sigma^+$ [] when hole appears to the right of the closest ancestor node labelled with  $\triangleright$ 

Note: different from traditional notion of polarity in display calculi which is relative to the top-most "turnstile"

Beware: our deep inference is not Guglielmi's Deep Inference!

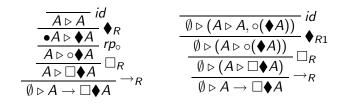
# Structural (Propagation) Rules in DBiKt

$$\frac{\Sigma^{-}[A, (A, X \triangleright Y)]}{\Sigma^{-}[A, X \triangleright Y]} \triangleright_{L1} \quad \frac{\Sigma^{+}[(X \triangleright Y, A), A]}{\Sigma^{+}[X \triangleright Y, A]} \triangleright_{R1} \quad \frac{\Sigma[(X \triangleright Y, A), W \triangleright A, Z]}{\Sigma[(X \triangleright Y), W \triangleright A, Z]} \triangleright_{R2}$$

$$\frac{\Sigma^{-}[A, \bullet(\Box A, X)]}{\Sigma^{-}[\bullet(\Box A, X)]} \Box_{L1} \quad \frac{\Sigma^{+}[A, \circ(\blacklozenge A, X)]}{\Sigma^{+}[\circ(\diamondsuit A, X)]} \blacklozenge_{R1} \quad \frac{\Sigma[\blacksquare A, X \triangleright \bullet(A \triangleright Y), Z]}{\Sigma[\blacksquare A, X \triangleright \bullet Y, Z]} \blacksquare_{L2}$$

rules are polarity dependent : positive, negative or simple contexts contraction: built into most rules i.e. premises contain conclusions propagation: of formulae when read backwards create structure: when read backwards proof search: must tame the application of these rules

#### Example of Shallow and Deep Derivations



Shallow

Deep

 $\begin{array}{ccc} & \rightarrow_r \text{ displays } A & & \rightarrow_r \text{ nests } A \\ \Box_r \text{ rewrites displayed } \Box A & & \Box_r \text{ rewrites nested } \Box A \\ r \rho_\circ \text{ displays } \blacklozenge A & & \blacklozenge_{R_1} \text{ deeply propagates } A \end{array}$ 

Equi-derivability between LBiKt and DBiKt

Right to left:

Theorem 4.1: every rule of DBiKt is derivable in LBiKt

Left to right: many technical proof-theoretic lemmas

Lemmas 4.2-4.6: LBiKt structural rules are DBiKt-admissible (if premisses derivable then so is conclusion) Logical: LBiKt rules mimicked (shallowly) in DBiKt (deep rules can always be applied shallowly)

# Modularity of **DBiKt** for its fragments

Purely modal nested sequent contains no occurrences of • nor its formula translates ■ and ◆

DInt: sub-system of DBiKt containing *id*, logical rules for intuitionistic connectives, and propagation rules for ▷

**DINTK**: is **DINT** plus the deep introduction rules for  $\Box$  and  $\Diamond$ , and the propagation rules  $\Box_{L2}$  and  $\Diamond_{R2}$ 

**DBInt**: is **DInt** plus the deep introduction rules for  $-\!\!<$ 

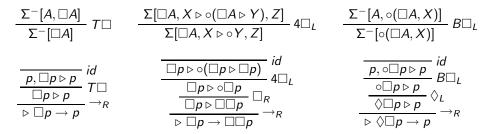
Modularity: for an Int/Bilnt/IntK formula A, the sequent  $\emptyset \triangleright A$  is **DInt/DBInt/DIntK**-derivable iff  $\emptyset \triangleright A$  is **DBiKt**-derivable

**Proof:** the only rules that create • upwards are  $\blacklozenge_L$  and  $\blacksquare_R$ . Thus in every **DBiKt**-derivation  $\pi$  of an IntK formula, the internal sequents in  $\pi$  are purely modal, so  $\pi$  is also a **DIntK**-derivation

## Syntactic Extensions

Evald: 
$$(\Diamond A \to \Box B) \to \Box (A \to B)$$
  $(\blacklozenge A \to \blacksquare B) \to \blacksquare (A \to B)$   
$$\frac{X \triangleright \bullet Y \triangleright \bullet Z}{X \triangleright \bullet (Y \triangleright Z)} \bullet \triangleright_R \qquad \frac{X \triangleright \circ Y \triangleright \circ Z}{X \triangleright \circ (Y \triangleright Z)} \circ \triangleright_R$$

Axiomatic Extensions

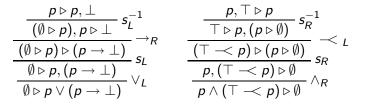


Note: do not correspond to reflexivity, transitivity, symmetry of  $R_{\Box}$ 

Classical Collapse Via Structural Rules

$$\frac{X_1, X_2 \triangleright Y_1, Y_2}{(X_1 \triangleright Y_1), X_2 \triangleright Y_2} s_L^{-1} \qquad \frac{X_1, X_2 \triangleright Y_1, Y_2}{(X_1 \triangleright Y_1, (X_2 \triangleright Y_2))} s_R^{-1}$$

The law of the excluded middle and the law of (dual-)contradiction can then be derived as shown below:



Proof Search Using the Deep System

Backward proof search strategy: proceeds in three stages: saturation, propagation and *realisation* 

Saturation phase: applies (backwards) the "static rules" (i.e. those that do not create extra structural connectives) until further backward application do not lead to any progress

Propagation phase: propagates formulaes across different structural connectives

Realisation phase: applies the "dynamic rules" (i.e., those that create new structural connectives, e.g.,  $\rightarrow_R$ )

Caveat: we do not yet have termination or completeness

Termination: of our proof search procedure Expressivity: what class of axiomatic extensions can we capture Implementation: extend our previous prover for Bilnt to BiKt

#### Rauszer's Axioms for Bilnt

$$(A \to B) \to ((B \to C) \to (A \to C))$$
 (1)

$$A \rightarrow A \lor B$$
 (2)

$$B \rightarrow A \lor B$$
 (3)

$$(A \to C) \to ((B \to C) \to ((A \lor B) \to C))$$
 (4)

$$(A \land B) \to A$$
 (5)

$$A \wedge B \rightarrow B$$
 (6)

$$(C \to A) \to ((C \to B) \to (C \to (A \land B)))$$
 (7)

(

$$(A \to (B \to C)) \to ((A \land B) \to C)$$
(8)

$$((A \land B) \to C) \to (A \to (B \to C))$$
 (9)

$$A \to (B \lor (A \prec B)) \tag{10}$$

$$(A \to B) \to (\neg B \to \neg A)$$
 (11)

$$(A \longrightarrow B) \longrightarrow \sim (A \longrightarrow B) \tag{12}$$

$$((A \longrightarrow B) \longrightarrow C) \rightarrow (A \longrightarrow (B \lor C))$$
(13)

$$\neg (A \longrightarrow B) \longrightarrow (A \longrightarrow B)$$
 (14)

$$(A \to (B \prec B)) \to \neg A \tag{15}$$

$$\neg A \rightarrow (A \rightarrow (B \rightarrow B))$$
 (16)

$$((B \rightarrow B) \rightarrow A) \rightarrow A$$
 (17)

$$\sim A \rightarrow ((B \rightarrow B) \prec A)$$
 (18)

MP plus From A infer  $\neg \sim A$