Semantic characterization of Kracht formulas

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Our basic language

$\mathcal{L}f_\Lambda$

- Λ is a set of indexes
- $\mathcal{L}f_\Lambda$ is the classical first order language with binary predicates indexed by Λ and equality;

Restricted quantifiers

$$(\forall x_i \rhd_\lambda x_j)A \equiv \forall x_i(x_j R_\lambda x_i \to A)$$

$$(\exists x_i \rhd_\lambda x_j)A \equiv \exists x_i(x_j R_\lambda x_i \wedge A)$$

Relational compositions

- if $\alpha = \lambda_1 \dots \lambda_n$ then $R^\alpha = R_{\lambda_1} \circ \dots \circ R_{\lambda_n}$;
- if α is empty then $R^\alpha = \mathit{equality}$.

Propositional constants

- $\top \equiv u = u$;
- $\bot \equiv u \neq u$.

$\mathcal{R}f_{\wedge}$

- $\bot, \top, xR^{\alpha}y, (x = y)$;
- $\wedge, \vee$;
- $(\forall x \rhd_{\lambda} y), (\exists x \rhd_{\lambda} y)$.

Inherently universal variables

Clean formulas

A formula A is clean $\iff$ there is no 2 different quantifiers binding the same variable.

Inherently universal variables

A variable x in a clean formula A, is inherently universal if either
x is free, or
x is bound by a universal quantifier which is not in the scope of any existential quantifier.

Kracht formulas

Kracht formulas

$A(x_0)$ is called a Kracht formula if

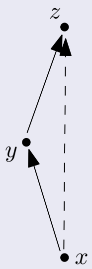
- $A(x_0) \in \mathcal{R}f_{\wedge}$;
- $A(x_0)$ has a single free variable x_0;
- in every subformula of $A(x_0)$ of the form $xR^{\alpha}y$ at least one of x and y is inherently universal.

Kracht's Theorem

$A(x_0)$ is a Kracht formula $\Longleftrightarrow$ $A(x_0)$ locally corresponds to a Sahlqvist formula (in a modal language with a few unary modalities).

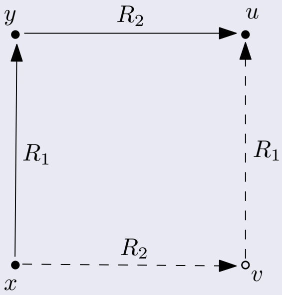
Examples of Kracht formulas

First-order equivalents of basic modal axioms



transitivity

$(\forall y \rhd x)(\forall z \rhd y) xRz$



(left) commutativity

$(\forall y \rhd_1 x)(\forall u \rhd_2 y)(\exists v \rhd_2 x) vR_1u$

Examples of Kracht formulas

However, Kracht formulas may contain an arbitrary long quantifier alternation.

$$(\forall x_1 \triangleright x_0)(\exists x_2 \triangleright x_1)(\forall x_3 \triangleright x_2) \ldots (\exists x_{n-1} \triangleright x_{n-2})(\forall x_n \triangleright x_{n-1})(x_1 R^\alpha x_n)$$

Some observations on Kracht formulas

Kracht formulas (another version)

- in every subformula of $A(x_0)$ of the form $xR^{\alpha}y$ x is inherently universal.

If y is inherently universal while x is not, then
$xR^{\lambda_1 \dots \lambda_n}y \equiv (\exists v_1 \rhd_{\lambda_1} x)(\exists v_2 \rhd_{\lambda_2} v_1) \dots (\exists v_n \rhd_{\lambda_n} v_{n-1})(y = v_n)$.

Some observations on Kracht formulas (II)

Kracht formulas seem do not have a prenex form

$$((\forall y \triangleright r)(\forall z \triangleright r)(yRz)) \wedge ((\exists v \triangleright r)\top)$$

$$\not\equiv$$

$$(\forall y \triangleright r)(\forall z \triangleright r)(yRz \wedge (\exists v \triangleright r)\top);$$

$$(\exists v \triangleright r)(\top \wedge (\forall y \triangleright r)(\forall z \triangleright r)(yRz))$$

is not a Kracht formula

Algorithmic problem

Given a first order formula determine if it is equivalent to a Kracht formula.

Undecidable (Chagrov, Zakharyashev, 1992; Chagrov, Chagrova, 2005).

Its partial cases

Learn how to prove that this or that that fixed first-order formula is not equivalent to a Kracht formula.

Generalizations of Sahlqvist theorem

- V. Goranko, D. Vakarelov, 2000 — 2006;
- E. Zolin, 2005.

Well known argument

The formula $(\Diamond\Diamond p \to p) \wedge (\Box\Diamond p \to \Diamond\Box p)$ is not equivalent to a Sahlqvist formula since it is not locally definable while Sahlqvist formulas are.

Approach of V. Goranko and D. Vakarelov

Problem:

Generalized Sahlqvist formula

$$D_2 = p \wedge \Box(\Diamond p \to \Box q) \to \Diamond\Box\Box q$$

locally corresponds to

$$\exists y \left(xRy \wedge \forall z \left(yR^2 z \to z \in R(R(x) \cap R^{-1}(x)) \right) \right).$$

Prove that the first formula not equivalent to any Sahlqvist formula, or that the second formula is not equivalent to any Kracht formula.

Approach of V. Goranko and D. Vakarelov

Solution: a-persistence

- A general frame $(W, (R_\lambda : \lambda \in \Lambda), \mathcal{A})$ is called *ample* if for any $w \in W$ for any $\alpha \in \Lambda^*$ $R^\alpha(w) \in \mathcal{A}$.
- A modal formula ϕ is *a*-persistent if for any ample general frame $F = (W, (R_\lambda : \lambda \in \Lambda), \mathcal{A})$ if $F \models \phi$ then $(W, (R_\lambda : \lambda \in \Lambda)) \models \phi$.
- Every Sahlqvist formula is *a*-persistent.
- The formula D_2 is not *a*-persistent.

Classical characterization theorems

Theorem

A class of frames is elementary iff it is closed under ultrapowers and elementary equivalence.

Theorem

A first-order formula is equivalent to a positive formula if it is preserved under model homomorphisms.

Theorem

A first-order formula is equivalent to an existential formula iff it is preserved under model extensions.

Theorem (van Benthem)

A first order formula (in a language with binary and unary predicates) is equivalent to a standard translation of a modal formula iff it is preserved under bisimulation.

Syntactic class of first-order formulas

$\Downarrow$

Truth-preserving relation between models

$\Downarrow$

Semantic characterization

Consider two models
$M = (W^M, (R^M_\lambda : \lambda \in \Lambda))$ and $N = (W^N, (R^N_\lambda : \lambda \in \Lambda))$.

Definition

ϕ distinguishes M from N if $M \models \phi$ and $N \not\models \phi$.

When M is distinguishable from N

by any first-order formula?	by a Kracht formula?
classical Eurenfeucht-Fraïssé game	?

Classical Eurenfeucht-Fraïssé game

- two players ($\forall$ and $\exists$) play over a pair of models M and N;
- the number of rounds is annonced by $\forall$ in the beginning of the game;
- a position is a pair of k-tuples $\bar{a} = (a_1, \dots, a_k), a_i \in W^M$ and $\bar{b} = (b_1, \dots, b_k), b_i \in W^N$.
- a position is favorable to $\exists$ if for any $\lambda \in \Lambda, 1 \leq i, j \leq k$

$$a_i R^M_\lambda a_j \iff b_i R^N_\lambda b_j$$

.

Modification of Eurenfeucht-Fraïssé game

- Kracht formulas have a single free variable $\Longrightarrow$ two players ($\forall$ and $\exists$) play over a pair of models with extinguished points $M_\circ = (M, x_0^M)$, $x_0^M \in W^M$ and $N_\circ = (N, x_0^N)$,$x_0^N \in W^N$;
- in Kracht formulas all quantifiers are restricted $\Longrightarrow$ a position is a triple (T, m, n) where $T = (W^T, (R_\lambda^T : \lambda \in \Lambda))$ is a tree with a root x_0, $W^T = \{x_0, \ldots, x_k\}$ and $m : T \to M$ and $n : T \to N$ are monotone mappings (i.e. $x_i R_\lambda^T x_j$ implies $m(x_i) R_\lambda^M m(x_j)$), sending x_0 respectively to x_0^M and x_0^N;

- a position is favorable to $\exists$ if for any $\alpha \in \Lambda^*, 1 \leq i, j \leq k$

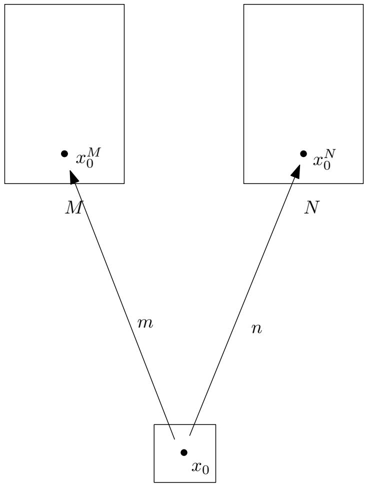
$$m(x_i)(R^M)^\alpha m(x_j) \Longrightarrow n(x_i)(R^N)^\alpha n(x_j),$$

  if x_i was played before first model alternation.
- the game may be either finite or infinite.

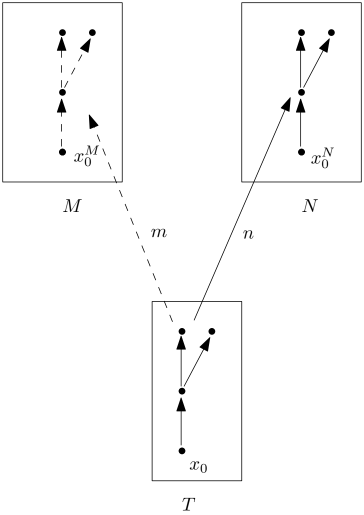
- Initial position: T has a single point x_0, m and n send it to x_0^M and x_0^N.
- Round 0. $\forall$ constructs $T_0 = (\{x_0, \dots, x_l\}, (R_\lambda : \lambda \in \Lambda))$ and a monotone $n_0 : T_0 \to N$. Then $\exists$ must answer with a monotone $m_0 : T_0 \to M$.
- Round i. $\forall$ adds to T_{i-1} a new leaf, and obtains T_i. Then he chooses either M or N and extends respectively m_{i-1} or n_{i-1} to T_i, obtaining m_i or n_i. After this, $\exists$ must extend the rest mapping, and a new position (T_i, m_i, n_i) is obtained.
- The game is won by $\exists$ if for any position (T_i, m_i, n_i) $\alpha \in \Lambda^*, 0 \le i \le l, 0 \le j \le k$

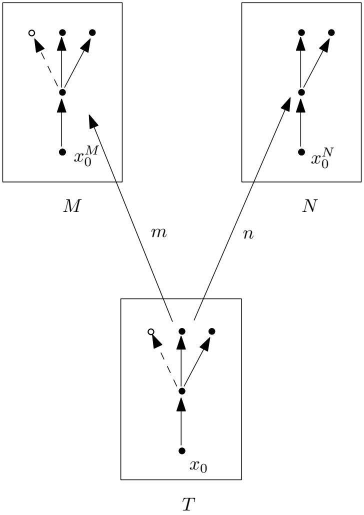
$$m(x_i)(R^M)^\alpha m(x_j) \implies n(x_i)(R^N)^\alpha n(x_j).$$

Initial position

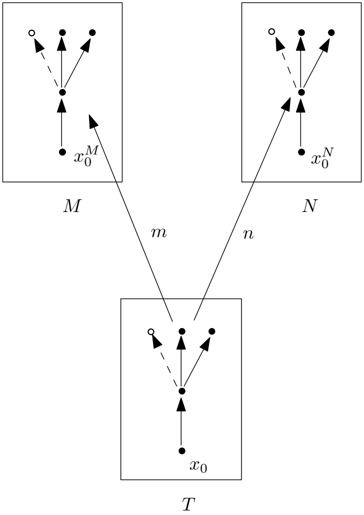


Round 0

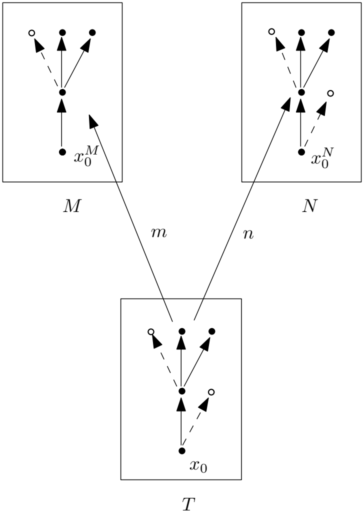



x_0^M
x_0^N
M
N
m
n
x_0
T

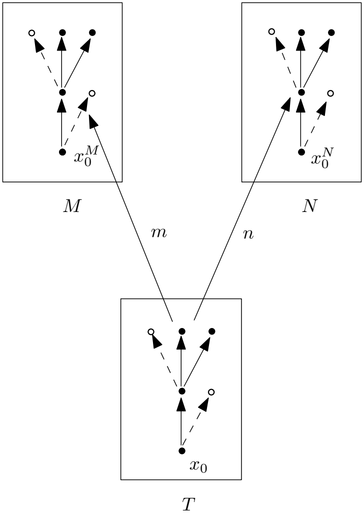
Round 1 (∃)



Round 2 (∀)



Round 2 (∃)



Kracht-reducibility in terms of games

Theorem

$\forall$ has a winning strategy in a finite version of such a game iff $M_\circ$ is distinguishable from $N_\circ$ by a Kracht formula

Definition

If $\exists$ has a winning strategy in an infinite game, we say that $M_\circ$ is Kracht-reducible to $N_\circ$, in symbols:

$$M_\circ \ggg N_\circ.$$

Remark

Winning strategy in an infinite game is a kind of (bi)simulation.

Kracht-reducibility in terms of simulations

Consider two $\mathcal{L}f_\Lambda$-structures $M = (W^M, (R^M_\lambda : \lambda \in \Lambda))$ and $N = (W^N, (R^N_\lambda : \lambda \in \Lambda))$, a tree $T = (W^T, (R^T_\lambda : \lambda \in \Lambda))$, monotonic mappings $m : T \to M$ and $n : T \to N$.

Definition

A relation $Z \subseteq W^M \times W^N$ is called a Kracht-simulation if Z satisfies the following conditions:

(KB1) For every $t \in W^T$, $(m(t), n(t)) \in Z$;

(KB2) For any $x^M \in W^M$, $x^N \in W^N$, $t \in T$, for arbitrary sequence $\alpha \in \Lambda^*$ if $(x^M, x^N) \in Z$ and $m(t)(R^M)^\alpha x^M$, then $n(t)(R^N)^\alpha x^N$.

Kracht-reducibility in terms of simulations

(KB3) For any points $x^M \in W^M$ and $x^N \in W^N$ such that $(x^M, x^N) \in Z$ if there exists a point $(x')^M \in R^M_\lambda(x^M)$, then there exists a point $(x')^N \in R^N_\lambda(x^N)$ such that $(x'^M, x'^N) \in Z$.

(KB4) For any points $x^M \in W^M$ and $x^N \in W^N$ such that $(x^M, x^N) \in Z$, if there exists a point $(x')^N \in R^N_\lambda(x^N)$, then there exists a point $(x')^M \in R^M_\lambda(x^M)$, such that $(x'^M, x'^N) \in Z$.

In this case we say that the triple (M, T, m) is Kracht-reducible to (N, T, n) by Z, in symbols: $(M, T, m) \gg_Z (N, T, n)$.

Kracht-reducibility in terms of simulations

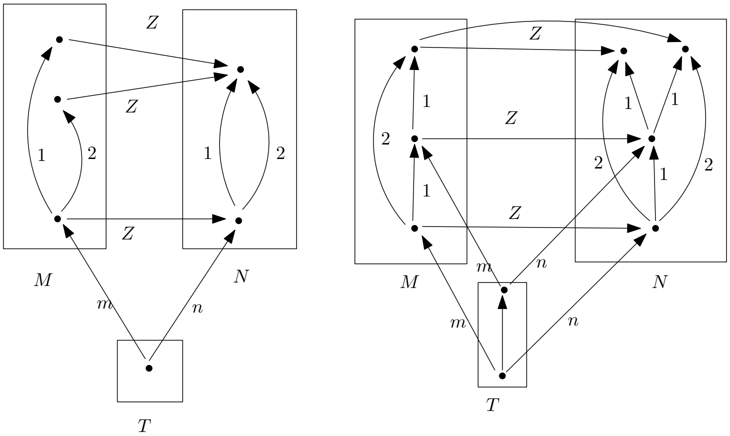


Figure: Some examples of Kracht-simulations.

Kracht-reducibility in terms of simulations

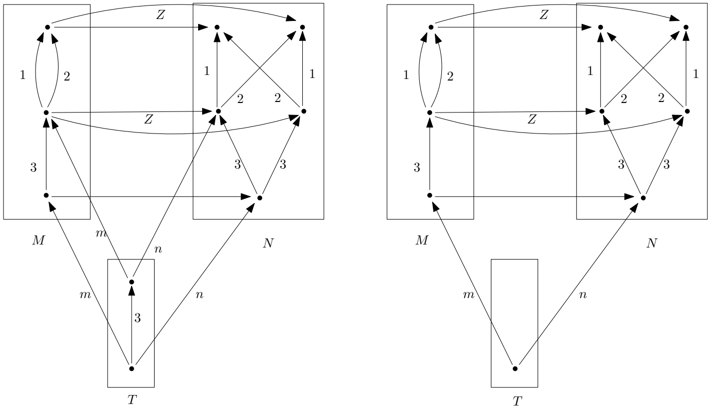


Figure: The left picture is not a Kracht-simulation while the right picture is.

Kracht-reducibility in terms of simulations

Definition

$M_\circ$ is Kracht-reducible to $N_\circ$ (notation: $M_\circ \ggg N_\circ$) if for any tree $T = (W^T, (R^T_\lambda : \lambda \in \Lambda), x_0)$ for all monotonic mappings $n : T \to N$, sending x_0 to x_0^N, there exists a monotonic mapping $m : T \to M$, sending x_0 to x_0^M, and a relation $Z \subseteq W^M \times W^N$ such that $(M, T, m) \gg_Z (N, T, n)$.

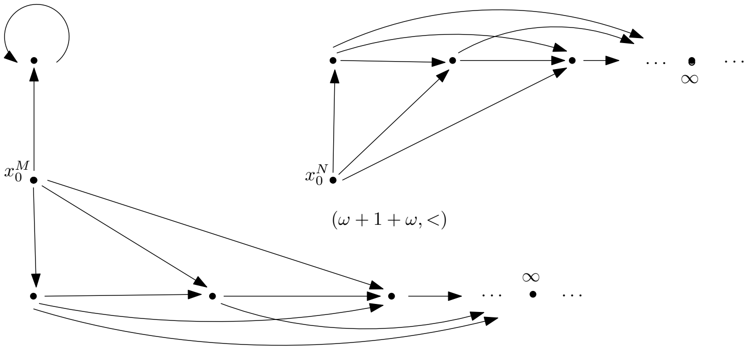
Definition

$A(x_0) \in \mathcal{L}f_\Lambda$ is preserved under Kracht-reducibility if $(M, x_0^M) \ggg (N, x_0^N)$ and $M \models A(x_0^M)$ implies $N \models A(x_0^N)$.

Theorem

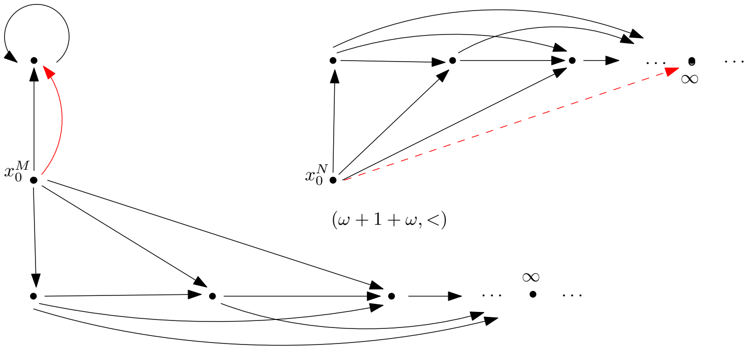
$A(x_0) \in \mathcal{L}f_\Lambda$ is equivalent to a Kracht formula iff it is preserved under Kracht-reducibility.

The formula $RS(x) = (\exists y \triangleright x)yRy$.



A trivial example

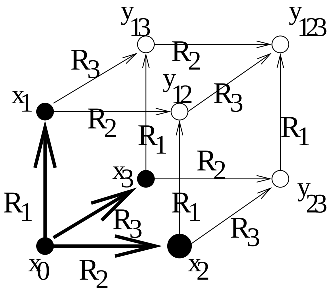
The formula $RS(x) = (\exists y \triangleright x) yRy$.



"Cubic" formula

All 3-modal frames of the form $F_1 \times F_2 \times F_3$, satisfy

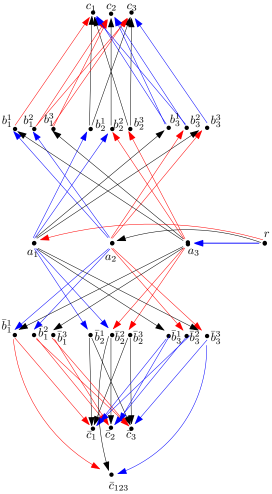
$$fc(x_0) = \forall x_1 \forall x_1 \forall x_3 (x_0 R_1 x_1 \wedge x_0 R_2 x_2 \wedge x_0 R_3 x_3 \rightarrow \exists y_{12} \exists y_{13} \exists y_{23} \exists y_{123} (x_1 R_2 y_{12} \wedge x_1 R_3 y_{13} \wedge x_2 R_1 y_{12} \wedge x_2 R_3 y_{23} \wedge x_3 R_1 y_{13} \wedge x_3 R_2 y_{23} \wedge y_{23} R_1 y_{123} \wedge y_{13} R_2 y_{123} \wedge y_{12} R_3 y_{123}))$$



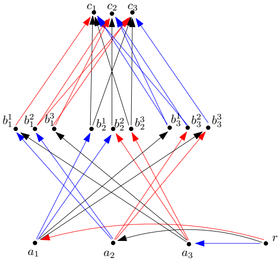
This property is locally first-order definable by a generalized Sahlqvist formula

$$cub_1 = [\Diamond_1(\Box_2 p_{12} \wedge \Box_3 p_{13}) \wedge \Diamond_2(\Box_1 p_{21} \wedge \Box_3 p_{23}) \wedge \Diamond_3(\Box_1 p_{31} \wedge \Box_2 p_{32}) \wedge \Box_1\Box_2(p_{12} \wedge p_{21} \to \Box_3 q_3) \wedge \Box_1\Box_3(p_{13} \wedge p_{31} \to \Box_2 q_2) \wedge \Box_2\Box_3(p_{23} \wedge p_{32} \to \Box_1 q_1)] \to \Diamond_1\Diamond_2\Diamond_3(q_1 \wedge q_2 \wedge q_3).$$

This property is not equivalent to any Kracht formula.



M



N

This characterization can be easily extended to generalized Kracht formulas.

(KB2') For any $x^M \in W^M$, $x^N \in W^N$, $t_1, \ldots, t_l \in T$, for all safe expressions $S(t_1, \ldots, t_l)$ if $(x^M, x^N) \in Z$ and $x^M \in S(m(t_1), \ldots, m(t_l))$, then $x^N \in x^M \in S(m(t_1), \ldots, m(t_l))$.

Questions

Question

Does modal definability of a first-order formula, and *a*-persinstance of its modal counterpart imply its equivalence to a Kracht formula?

Question

Take a class of first-order formulas C. It defines some truth-preserving relation between models: put $M \ggg N$ if for all $\phi \in C$ if $M \models \phi$ then $N \models \phi$.
When we can claim that any first-order formula ψ is equivalent to a formula from C iff ψ is preserved under $\ggg$?