WEAK INTERPOLATION PROPERTY over THE MINIMAL LOGIC

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Abstract

Weak interpolation property WIP in extensions of Johansson's minimal logic J is investigated. A weak version of the joint consistency property equivalent to WIP is found. It is proved that the weak interpolation property is decidable over J.

Interpolation theorem proved by W.Craig in 1957 for the classical first order logic was a source of a lot of research results devoted to interpolation problem in classical and non-classical logical theories. Now interpolation is considered as a standard property of logics and calculi like consistency, completeness and so on. For the intuitionistic predicate logic and for the predicate version of Johansson's minimal logic the interpolation theorem was proved by K.Schütte (1962).

In this paper we consider a variant of the interpolation property in the minimal logic and its extension. The minimal logic introduced by I.Johansson (1937) has the same positive fragment as the intuitionistic logic but has no special axioms for negation. In contrast to the classical and intuitionistic logics, the minimal logic admits non-trivial theories containing some proposition together with its negation.

The original definition of interpolation admits different analogs which are equivalent in the classical logic but are not equivalent in other logics. It is known that in classical theories the interpolation property is equivalent to the joint consistency RCP, which arises from the joint consistency theorem proved by A.Robinson (1956) for the classical predicate logic. It was proved by D. Gabbay (1981) that in the intuitionistic predicate logic the full version of RCP does not hold. But some weaker version of RCP is valid, and this weaker version is equivalent to CIP in all superintuitionistic predicate logics.

In this paper we concentrate on the weak interpolation property WIP introduced in M2005. We prove that WIP is equivalent to a version JCP of Robinson consistency property in all extensions of the minimal logic.

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In this paper we concentrate on the weak interpolation property WIP introduced in M2005. We prove that WIP is equivalent to a version JCP of Robinson consistency property in all extensions of the minimal logic.

In M2005 we noted that all propositional superintuitionistic logics have WIP, although it does not hold for superintuitionistic predicate logics. Since only finitely many propositional superintuitionistic logics possess CIP (M77) , WIP and JCP are not equivalent to CIP and RCP over the intuitionistic logic. Also WIP is non-trivial in propositional extensions of the minimal logic.

We define a J-logic Gl and state that the problem of weak interpolation in J-logics is reducible to the same problem over Gl. Algebraic criteria for WIP in J-logics are given.

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Interpolation and joint consistency

If $\mathbf{p}$ is a list of non-logical symbols, let $A(\mathbf{p})$ denote a formula whose all non-logical symbols are in $\mathbf{p}$, and $\mathcal{F}(\mathbf{p})$ the set of all such formulas.

Let L be a logic, $\vdash_L$ deducibility relation in L. Suppose that $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ are disjoint lists of non-logical symbols, and $A(\mathbf{p}, \mathbf{q}, x)$, $B(\mathbf{p}, \mathbf{r})$ are formulas. *The Craig interpolation property CIP* and *the deductive interpolation property IPD* are defined as follows:

CIP. If $\vdash_L A(\mathbf{p}, \mathbf{q}) \to B(\mathbf{p}, \mathbf{r})$, then there exists a formula $C(\mathbf{p})$ such that $\vdash_L A(\mathbf{p}, \mathbf{q}) \to C(\mathbf{p})$ and $\vdash_L C(\mathbf{p}) \to B(\mathbf{p}, \mathbf{r})$.

IPD. If $A(\mathbf{p},\mathbf{q}) \vdash_L B(\mathbf{p},\mathbf{r})$, then there exists a formula $C(\mathbf{p})$ such that $A(\mathbf{p},\mathbf{q}) \vdash_L C(\mathbf{p})$ and $C(\mathbf{p}) \vdash_L B(\mathbf{p},\mathbf{r})$.

In M2005 *the weak interpolation property* was introduced:

WIP. If $A(\mathbf{p},\mathbf{q}), B(\mathbf{p},\mathbf{r}) \vdash_L \bot$, then there exists a formula $A'(\mathbf{p})$ such that $A(\mathbf{p},\mathbf{q}) \vdash_L A'(\mathbf{p})$ and $A'(\mathbf{p}), B(\mathbf{p},\mathbf{r}) \vdash_L \bot$.

In all extensions of the minimal logic we have

$$CIP \Leftrightarrow IPD \Rightarrow WIP.$$

In the classical predicate logic CIP is equivalent to the Robinson consistency property

RCP. Let T_1, T_2 be two consistent L-theories in the languages $\mathcal{L}_1, \mathcal{L}_2$ respectively. If $T_1 \cap T_2$ is a complete L-theory in the common language $\mathcal{L}_1 \cap \mathcal{L}_2$, then $T_1 \cup T_2$ is L-consistent.

The same equivalence holds in all classical modal logics.

Let L be any axiomatic extension of the minimal logic. An *open L-theory* of a language $\mathcal{L}$ is a set $T \subseteq \mathcal{L}$ closed with respect to $\vdash_L$. An open L-theory T is *L-consistent* if it does not contain $\bot$. An L-theory T of the language $\mathcal{L}$ is *complete* if $A \in T$ or $\neg A \in T$ for any sentence $A \in \mathcal{L}$.

We define *the joint consistency property*

JCP. Let T_1, T_2 be two L-consistent open L-theories in the languages $\mathcal{L}_1, \mathcal{L}_2$ respectively, and $T_1 \cap T_2$ be complete in the common language $\mathcal{L}_0 = \mathcal{L}_1 \cap \mathcal{L}_2$. Then $T_1 \cup T_2$ is L-consistent.

Theorem

For any (predicate or propositional) extension L of the minimal logic, WIP is equivalent to JCP.

Propositional J-logics

In this section we study propositional J-logics.

In M77 a description of all propositional superintuitionistic logics with interpolation property was obtained. There are only finitely many superintuitionistic logics with this property. All positive logics with the interpolation property were described in M2003, where a study of this property was initiated for extensions of Johansson's minimal logic too. The minimal logic and the intuitionistic logic have the Craig interpolation property (Schütte 1962).

The language of the logic J contains $\&, \vee, \rightarrow, \bot, \top$ as primitive; negation is defined by $\neg A = A \rightarrow \bot$; $(A \leftrightarrow B) = (A \rightarrow B)\&(B \rightarrow A)$. A formula is said to be *positive* if contains no occurrences of $\bot$. The logic J can be axiomatized by the calculus, which has the same axiom schemes as the positive intuitionistic calculus Int^+, and the only rule of inference is modus ponens [6]. By a J-*logic* we mean an arbitrary set of formulas containing all the axioms of J and closed under modus ponens and substitution rules. We denote

$$\mathrm{Int} = \mathrm{J} + (\bot \rightarrow p),\ \mathrm{Cl} = \mathrm{Int} + (p \vee \neg p),\ \mathrm{Neg} = \mathrm{J} + \bot.$$

A logic is *non-trivial* if it differs from the set of all formulas. A J-logic is *superintuitionistic* if it contains the intuitionistic logic Int, and *negative* if contains the logic Neg; L is *paraconsistent* if contains neither Int nor Neg. One can prove that a logic is negative if and only if it is not contained in Cl. For any J-logic L we denote by $E(L)$ the family of all J-logics containing L.

It was proved in [7] that all propositional superintuitionistic logics possess the weak interpolation property WIP. Evidently, all negative logics also have this property. We prove

Theorem

Any propositional J*-logic containing* $\mathrm{J} + (\bot \vee (\bot \to p))$ *possesses WIP.*

Reduction of WIP

Theorem 2 can not be extended to all J-logics. The picture changes when we turn to extensions of the logic

$$\mathrm{Gl} = \mathrm{J} + (p \vee (p \rightarrow \bot)) = \mathrm{J} + (p \vee \neg p).$$

It was proved in [10] that this logic has CIP. There are extensions of Gl without WIP. Here we prove that the problem of weak interpolation is reducible to the same problem in extensions of Gl.

Consider extensions of Gl in more detail. We note that for any extension of Gl the following analog of Glivenko's theorem holds.

Lemma

For any J*-logic L and any formula A:*

$$L + (p \vee \neg p) \vdash \neg A \iff L \vdash \neg A.$$

We prove that the problem of weak interpolation in J-logics can be reduced to the same problem over Gl.

Theorem

For any J*-logic L, the logic L has WIP if and only if* $L + (p \vee \neg p)$ *has WIP.*

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Weak interpolation and weak amalgamation

For extensions of the minimal logic the algebraic semantics is built with using so-called J-*algebras*, i.e. algebras $\mathbf{A} =< A; \&, \vee, \rightarrow, \bot, \top >$ satisfying the conditions:

$< A; \&, \vee, \rightarrow, \bot, \top >$ is a lattice with respect to $\&, \vee$ having a greatest element $\top$, where

$z \leq x \rightarrow y \iff z \& x \leq y$,

$\bot$ is an arbitrary element of A.

A formula A is said to be *valid* in a J-algebra $\mathbf{A}$ if the identity $A = \top$ is satisfied in $\mathbf{A}$.

Weak interpolation and weak amalgamation

For extensions of the minimal logic the algebraic semantics is built with using so-called J-*algebras*, i.e. algebras $\mathbf{A} =< A; \&, \vee, \rightarrow, \perp, \top >$ satisfying the conditions:

$< A; \&, \vee, \rightarrow, \perp, \top >$ is a lattice with respect to $\&, \vee$ having a greatest element $\top$, where

$z \leq x \rightarrow y \iff z \& x \leq y$,

$\perp$ is an arbitrary element of A.

A formula A is said to be *valid* in a J-algebra $\mathbf{A}$ if the identity $A = \top$ is satisfied in $\mathbf{A}$.

A J-algebra is called a *Heyting algebra* if $\perp$ is the least element of A, and a *negative algebra* if $\perp$ is the greatest element of A. A one-element J-algebra is said to be *degenerate*; it is the only J-algebra, which is both a negative algebra and a Heyting algebra. A J-algebra **A** is *non-degenerate* if it contains at least two elements; **A** is said to be *well connected* (or *strongly compact*) if for all $x, y \in$ **A** the condition $x \vee y = \top \Leftrightarrow (x = \top$ or $y = \top)$ is satisfied. An element a of **A** is called an *opremum of* **A** if it is the greatest among the elements of **A** different from $\top$. By B_0 we denote the two-element Boolean algebra $\{\perp, \top\}$.

In this section we find an algebraic equivalent of the weak interpolation property.
It is well known that the family of all J-algebras forms a variety, i.e. can be determined by identities. There exists a one-to-one correspondence between logics extending the logic J and varieties of J-algebras. If A is a formula and $\mathbf{A}$ is an algebra, we say that A is valid in $\mathbf{A}$ and write $\mathbf{A} \models A$ if the identity $A = \top$ is satisfied in $\mathbf{A}$. We write $\mathbf{A} \models L$ instead of $(\forall A \in L)(\mathbf{A} \models A)$.

To any logic $L \in E(\mathrm{J})$ there corresponds a variety

$$V(L) = \{\mathbf{A} | \mathbf{A} \models L\}.$$

Every logic L is characterized by the variety $V(L)$.
If $L \in E(\mathrm{Int})$, then $V(L)$ is a variety of Heyting algebras, and if $L \in E(\mathrm{Neg})$, then a variety of negative algebras.
Recall [9] that a J-logic has the Craig interpolation property if and only if $V(L)$ has the amalgamation property AP.

We recall necessary definitions.

Let V be a class of algebras invariant under isomorphisms. The class V has *the amalgamation property* if it satisfies the following condition AP for any algebras **A**,**B**,**C** in V:

(AP) if **A** is a common subalgebra of **B** and **C**, then there exist **D** in V and monomorphisms $\delta : \mathbf{B} \to \mathbf{D}$, $\varepsilon : \mathbf{C} \to \mathbf{D}$ such that $\delta(x) = \varepsilon(x)$ for all $x \in \mathbf{A}$.

A variety V of J-algebras is *weakly amalgamable* if it satisfies WAPJ: For all $\mathbf{B}, \mathbf{C} \in V$ with a common subalgebra $\mathbf{A}$, there are $\mathbf{D} \in V$ and homomorphisms $\delta : \mathbf{B} \to \mathbf{D}$ and $\varepsilon : \mathbf{C} \to \mathbf{D}$ such that $\delta(x) = \varepsilon(x)$ for all $x \in \mathbf{A}$, where $\bot \neq \top$ in $\mathbf{D}$ whenever $\bot \neq \top$ in $\mathbf{A}$.

Theorem

A J*-logic has WIP iff* $V(L)$ *has WAPJ.*

If $\mathbf{A} =< A; \&, \vee, \rightarrow, \bot, \top >$ is a negative algebra, we define a new J-algebra $\mathbf{C} = \mathbf{A}^{\Lambda}$ which arises from $\mathbf{A}$ by adding a new greatest element $\top$ so that $\bot_{\mathbf{C}} = \bot_{\mathbf{A}}$. Denote $\Lambda(L) = \{\mathbf{A}^{\Lambda} | \ \mathbf{A}^{\Lambda} \in V(L)\}$.

In the following theorem we formulate an algebraic criterion for WIP in J-logics.

Theorem

A J-logic L has WIP iff $\Lambda(L)$ has the amalgamation property.

Logics with WIP over Gl

For any negative logic L we define two logics $(L \uparrow \mathrm{Cl})$ and $(L \Uparrow \mathrm{Cl})$ [10]. The former is characterized by all J-algebras of the form $\mathbf{A}^{\Lambda}$, where $\mathbf{A} \in V(L_1)$, and the latter by J-algebras $\mathbf{A}^{\Lambda}$, where $\mathbf{A}$ is a finitely indecomposable algebra in $V(L)$. In [10] the following axiomatization was found:

$$L \uparrow \mathrm{Cl} = \mathrm{Gl} + \{\bot \to A |\ A \in L, \}$$

$$L \Uparrow \mathrm{Cl} = (L \uparrow \mathrm{Cl}) + ((\bot \to A \vee B) \to (\bot \to A) \vee (\bot \to B)).$$

We have $\mathrm{Gl} = \mathrm{Neg} \uparrow \mathrm{Cl}$. Every logic L' of the form $L \uparrow \mathrm{Cl}$ or $L \Uparrow \mathrm{Cl}$ is generated by $\Lambda(L')$.

In [M2003] all extensions of the logic Neg with CIP were found:

$$\mathrm{Neg},\ \mathrm{NC} = \mathrm{Neg} + (p \to q) \vee (q \to p),$$

$$\mathrm{NE} = \mathrm{Neg} + p \vee (p \to q),\ \mathrm{For} = \mathrm{Neg} + p.$$

Theorem

The logics Cl, $(\mathrm{NE} \uparrow \mathrm{Cl})$, $(\mathrm{NC} \uparrow \mathrm{Cl})$, $(\mathrm{Neg} \uparrow \mathrm{Cl})$, $(\mathrm{NE} \Uparrow \mathrm{Cl})$, $(\mathrm{NC} \Uparrow \mathrm{Cl})$, $(\mathrm{Neg} \Uparrow \mathrm{Cl})$ *have the finite model property and so are decidable. For each of these logics the variety* $V(L)$ *is generated by* $\Lambda(L)$.

Theorem

Let L be one of the following Gl*-logics:* Cl, (NE $\uparrow$ Cl), (NC $\uparrow$ Cl), (Neg $\uparrow$ Cl), (NE $\Uparrow$ Cl), (NC $\Uparrow$ Cl), (Neg $\Uparrow$ Cl). *Then L has CIP, and the classes $V(L)$ and $\Lambda(L)$ have the amalgamation property.*

Theorem

If L is a negative logic without CIP, then the logics $L \uparrow \mathrm{Cl}$ *and* $L \Uparrow \mathrm{Cl}$ *do not possess WIP.*

Corollary

There is a continuum of J*-logics with WIP and a continuum of* J*-logics without WIP.*

Description of logics with WIP over Gl and J

First we describe all logics with WIP over Gl. Denote

$$SL = \{\mathrm{For}, \mathrm{Cl}\} \cup \{(\mathrm{L}_1 \uparrow \mathrm{Cl}),\ (\mathrm{L}_1 \Uparrow \mathrm{Cl}) \mid \mathrm{L}_1 \in \{\mathrm{Neg}, \mathrm{NC}, \mathrm{NE}\}\}.$$

Theorem

A logic L over Gl has WIP iff $\mathrm{Neg} \cap L_0 \subseteq L \subseteq L_0$ for some L_0 in the list SL.

So the logics with WIP over Gl are divided into eight disjoint intervals.

Theorem

WIP is decidable over J*, i.e. there is an algorithm which, given a finite set Ax of axiom schemes, decides if the logic* $\mathrm{J} + Ax$ *has WIP.*

The following problems are still open:

1. Is CIP decidable over J?
2. How many J-logics possess CIP?

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