Pertinence Construed Modally

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A Simple Example (apology to Russell)

Let

- p: "Mars orbits the Sun"
- q: "a red teapot is orbiting Mars"

In Classical Logic

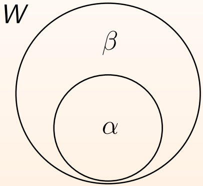
- $\neg p \wedge q \models q$ (disjunctive syllogism: $\neg p \wedge (p \vee q) \models q$)
- $\neg p \models \neg p \vee q$
- $\neg p \models \top$
- $\bot \models \neg p$ (ex contradictione quodlibet)
- $\models p \rightarrow (q \rightarrow p)$ (positive paradox)

But

Some notion of *relevance* or *pertinence*, should hold between premiss and consequence

Classical Logic: the Logic of ‘Complete Ignorance’

- $\alpha \models \beta$



Usually

Extra information expressed either as

- *Syntactic rules*, or as
- *Semantic constraints* on sets of sentences

Less Attractive Features of Traditional Relevance Logics [Avron, 1992]

- Conflation of $\models$ with $\rightarrow$ [Anderson and Belnap, 1975, 1992]
- Start with *proof theory*, then find a proper semantics
- Sometimes *metaphysical* ideas get admixed into the relevance endeavour
- Relevance logics traditionally pay scant attention to *contexts*

Outline

Modal Logic

Here: standard modal logics

- Propositional language
- Possible worlds semantics

Classes of models

- Sets of models we work with
- Determined by additional constraints
  - Axiom schemas (reflexivity, transitivity, etc.)
  - Global axioms (see later)

Here we are interested in the class of reflexive models

- Axiom schema $\Box\alpha \rightarrow \alpha$: Modal logic KT

Modal Logic

Local consequence

Definition

α *entails* β *in* $\mathscr{M} = \langle W, R, V \rangle$ (denoted $\alpha \models^{\mathscr{M}} \beta$) iff for every $w \in W$, if $w \Vdash^{\mathscr{M}} \alpha$, then $w \Vdash^{\mathscr{M}} \beta$.

Definition

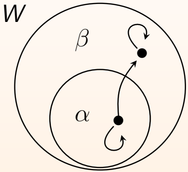
Let $\mathscr{C}$ be a class of models

- α *entails* β *in* $\mathscr{C}$ (denoted $\alpha \models^{\mathscr{C}} \beta$) iff $\alpha \models^{\mathscr{M}} \beta$ for every $\mathscr{M} \in \mathscr{C}$
- Validity and satisfiability in $\mathscr{C}$ defined as usual

When $\mathscr{C}$ is clear from the context, we write $\alpha \models \beta$ instead of $\alpha \models^{\mathscr{C}} \beta$

Pertinence in the Meta-Level

- The consequent β should not run wild



Definition

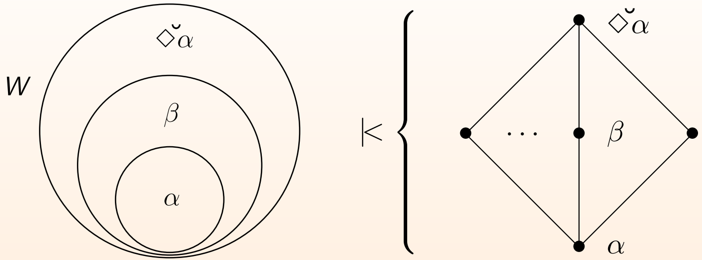
α pertinently entails β in $\mathscr{M}$ (denoted $\alpha \mathrel{|\!\!<^{\mathscr{M}}} \beta$) iff $\alpha \models^{\mathscr{M}} \beta$ and $\beta \models^{\mathscr{M}} \breve{\Diamond}\alpha$

Definition

α pertinently entails β in the class $\mathscr{C}$ of models (denoted $\alpha \mathrel{|\!\!<^{\mathscr{C}}} \beta$) iff for every $\mathscr{M} \in \mathscr{C}$, $\alpha \mathrel{|\!\!<^{\mathscr{M}}} \beta$

When $\mathscr{C}$ is clear from the context, we write $\alpha \mathrel{|\!\!<} \beta$ instead of $\alpha \mathrel{|\!\!<^{\mathscr{C}}} \beta$

Pertinence in the Meta-Level



- Clearly, $|\!<$ is *infra-modal*: if $\alpha \mid\!< \beta$, then $\alpha \models \beta$
- '$|\!<$' vs. '$\models$' like '$<$' vs. '$=$'

A Spectrum of Entailment Relations

- Only restriction on R: $id_W \subseteq R \subseteq W \times W$ (modal logic KT)
- The *minimum* (w.r.t. $\subseteq$) case: $R = id_W$
  - *maximum pertinence*: $\mathrel{|\!\!<} = \equiv$
- The *maximum* case: $R = W \times W$ (assume $\alpha \not\equiv \bot$, cf. later)
  - *minimum pertinence*: $\mathrel{|\!\!<} = \models$

Theorem

If the underlying modal logic is at least KT, *then* $\equiv \subseteq \mathrel{|\!\!<} \subset \models$

- Note how, psychologically speaking, with increased pertinence between premiss and consequence 'if' tends to drift in the direction of 'if and only if' [Johnson-Laird & Savary, 1999]

Properties of $\mid\!<$

- Decidability
  - Straightforward from definition
- Non-explosiveness
  - if $\bot \mid\!< \alpha$, then $\alpha \equiv \bot$

Theorem

Let $\alpha \mid\!\overset{\mathscr{C}}{<} \beta$. *Then if* $\models^{\mathscr{C}} \alpha \to \bot$, *then* $\models^{\mathscr{C}} \beta \to \bot$

- $\mid\!<$ is *paratrivial*
  - $\alpha \not\mid\!< \top$ in general
- $\mid\!<$ preserves *validities*

Theorem

$\top \mid\!< \alpha$ *iff* $\top \models \alpha$

Properties of $\mathrel{|\!\!<}$

- Disjunctive Syllogism: $(\neg\alpha \vee \beta) \wedge \alpha \not\mathrel{|\!\!<} \beta \quad (\beta \wedge \alpha \not\mathrel{|\!\!<} \beta)$
- $\mathrel{|\!\!<}$ does not satisfy contraposition
- $\mathrel{|\!\!<}$ does not satisfy the deduction theorem $\alpha \mathrel{|\!\!<} \beta$ iff $\top \mathrel{|\!\!<} \alpha \rightarrow \beta$
- Modus Ponens

$$\frac{\mathrel{|\!\!<} \alpha, \ \mathrel{|\!\!<} \alpha \rightarrow \beta}{\mathrel{|\!\!<} \beta}$$

- Non-Monotonicity: For $\mathrel{|\!\!<}$, the monotonicity rule fails:

$$\frac{\alpha \mathrel{|\!\!<} \beta, \ \gamma \models \alpha}{\gamma \mathrel{|\!\!<} \beta}$$

- Substitution of Equivalents
- Transitivity: If the underlying logic is at least S4

$$\frac{\alpha \mathrel{|\!\!<} \beta, \ \beta \mathrel{|\!\!<} \gamma}{\alpha \mathrel{|\!\!<} \gamma}$$

Pertinent Conditional

Definition

$\alpha \mathrel{\diamond\!\!\rightarrow} \beta \equiv_{\mathsf{def}} (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \breve{\Diamond}\alpha)$

Theorem

$\alpha \mathrel{|\!\!<} \beta$ *iff* $\mathrel{|\!\!<} \alpha \mathrel{\diamond\!\!\rightarrow} \beta$

Proposition

- $\mathrel{|\!\!<} \alpha \rightarrow (\beta \rightarrow \alpha)$ (*positive paradox*)
- $\mathrel{|\!\!\not<} \alpha \mathrel{\diamond\!\!\rightarrow} (\beta \mathrel{\diamond\!\!\rightarrow} \alpha)$
- $\alpha \mathrel{|\!\!\not<} \beta \mathrel{\diamond\!\!\rightarrow} \alpha$

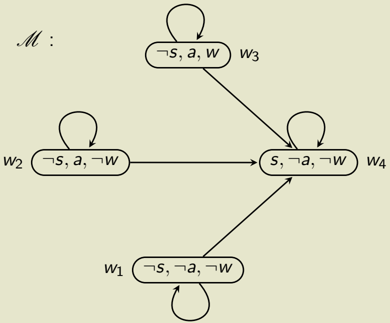
Pertinence and Causation

Example

- s: "the turkey is shot"; a: "it is alive"; w: "the turkey is walking"

  Background assumption:

$\mathcal{B} = \{w \rightarrow a, s \rightarrow \neg a, \Diamond s\}$



Question: Is α the pertinent *cause* of β?

- $\neg a \wedge \neg w \mid\!\!< \neg a$; $\neg a \wedge \neg w \mid\!\!\not< \neg w$
- $a \wedge \Box\neg s \mid\!\!\not< a$; $a \wedge \Box\Diamond s \mid\!\!< a$
- $s \mid\!\!\not< \neg a$; $s \vee \neg a \mid\!\!< \neg a$

General Setting

- Definition of a *weakening operator* $\bullet$
  - Weakening $\alpha \models \bullet\alpha$
  - Uniformity If $\alpha \models \beta$, then $\bullet\alpha \models \bullet\beta$
- Pertinence as *additional constraint* given by $\bullet$
  - $\alpha \mathrel{|\!\!<} \beta$ iff $\alpha \models \beta$ and $\beta \models \bullet\alpha$
- General pertinent entailment relation
  - Reflexivity $\alpha \mathrel{|\!\!<} \alpha$
  - Infraclassicality If $\alpha \mathrel{|\!\!<} \beta$, then $\alpha \models \beta$
  - Generalized Disjunction If $\alpha \mathrel{|\!\!<} \beta$ and $\gamma \mathrel{|\!\!<} \delta$, then $\alpha \vee \gamma \mathrel{|\!\!<} \beta \vee \delta$
  - Interpolation If $\alpha \models \beta$, $\beta \models \gamma$, and $\alpha \mathrel{|\!\!<} \gamma$, then $\alpha \mathrel{|\!\!<} \beta$ and $\beta \mathrel{|\!\!<} \gamma$
- Here we have seen one case: $\bullet = \breve{\Diamond}$

Conclusion

Contributions

- Semantic approach to the notion of pertinence
- Pertinence captured in a simple modal logic
- Whole spectrum of pertinent entailments, ranging between $\equiv$ and $\models$
- We restrict some paradoxes avoided by relevance logics
- $|\!<$ possesses other non-classical properties

Ongoing and Future Work

- Other infra-modal entailment relations
- Supra-modal entailment: *prototypical* and *venturous reasoning*
- Relationship with contexts such as *obligations*, *beliefs*, etc
- Pertinent subsumptions in Description Logics

More: http://krr.meraka.org.za

Thank you!