Future event logic - axioms and complexity

Hans van Ditmarsch
University of Sevilla, Spain

Tim French
University of Western Australia, Australia

Sophie Pinchinat
Universit de Rennes 1, France

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Informative Events

Consider a system that consists of a set of agents and a set of facts. The facts are known to be static so they do not change, although whether an agent \textit{knows} a proposition is true may change. An agent may experience an informative event where their uncertainty in the system is reduced. An informative event is any change that updates a model in such a way that it is consistent with at least one of the “possibilities” inherent in the original model. Examples of informative events include announcements (public or private), message passing systems and action models.
Example

Alice and Bob have both applied for a tenured lecturing position, and are waiting outside the Dean’s office to hear which one has won the position. The Dean asks Alice to come into the office. He tells her she has won the position and she leaves.

This is an informative event. However, from Bob’s point of view it has increased uncertainty. Previously he knew Alice did not know who had the job, but he considers it possible that she knows she has the job, that she knows she does not have the job, or that the Dean told her he has not yet made his decision.
Example

This graphic represents the effect of the informative event for Alice. Each circle represents a world where either Alice or Bob got the job, and an agent’s uncertainty between which world is the actual world is represented by the relations. The underlined world is the actual world.
Technical preliminaries

A finite set of agents $A$ and a countably infinite set of atoms $P$

**Structures**

$M = (S, R, V)$

- $S \ni s, t, \ldots$ a domain of states
- $R : A \rightarrow \mathcal{P}(S \times S)$ accessibility relation; write $R_a(s, t)$
- $V : P \rightarrow \mathcal{P}(S)$ a valuation

For $s \in S$, $M_s$ is a state or a pointed Kripke model.

**Bisimulation**

$M = (S, R, V)$ and $M' = (S', R', V')$. $\mathcal{R} \subseteq S \times S'$ is a bisimulation whenever $(s, s') \in \mathcal{R}$ if for all $a \in A$:

- atoms $s \in V(p)$ iff $s' \in V'(p)$ for all $p \in P$
- forth-$a$ $\forall t \in sR_a$, $\exists t' \in s'R'_a$ with $(t, t') \in \mathcal{R}$
- back-$a$ vice versa
Technical preliminaries

A finite set of agents $A$ and a countably infinite set of atoms $P$

Bisimulation

$M = (S, R, V)$ and $M' = (S', R', V')$. $\mathcal{R} \subseteq S \times S'$ is a \textit{bisimulation} whenever $(s, s') \in \mathcal{R}$ if for all $a \in A$:

- atoms: $s \in V(p)$ iff $s' \in V'(p)$ for all $p \in P$
- forth-$a$: $\forall t \in sR_a, \exists t' \in s'R'_a$ with $(t, t') \in \mathcal{R}$
- back-$a$: vice versa

Simulation

A relation that satisfies

- atoms: $s \in V(p)$ iff $s' \in V'(p)$ for all $p \in P$
- forth-$a$: for every $b \in A$

is a \textit{simulation}. In that case $M'_s$ is a \textit{simulation} of $M_s$, and $M_s$ is a \textit{refinement} of $M'_s$, and we write $M_s \preceq M'_s$. 
Technical preliminaries

A finite set of agents $A$ and a countably infinite set of atoms $P$

Simulation

A relation that satisfies

- atoms $s \in V(p)$ iff $s' \in V'(p)$ for all $p \in P$
- forth-$a$ for every $b \in A$

is a simulation. In that case $M_{s'}$ is a simulation of $M_s$, and $M_s$ is a refinement of $M_{s'}$, and we write $M_s \preceq M_{s'}$.

$a$-simulation ($a \in A$)

A relation that satisfies

- atoms
- forth-$a$ for every $b \in A$
- back-$a$ for every $b \in A - \{a\}$

is an $a$-simulation.

In that case $M_s$ is an $a$-refinement of $M_{s'}$, and we write $M_s \preceq_a M_{s'}$. 
Technical preliminaries
A finite set of agents $A$ and a countably infinite set of atoms $P$

**$a$-simulation ($a \in A$)**

A relation that satisfies

- atoms
- forth-$a$ for every $b \in A$
- back-$a$ for every $b \in A - \{a\}$

is an *$a$-simulation*.

In that case $M_s$ is an *$a$-refinement* of $M_{s'}$, and we write $M_s \leq_a M_{s'}$.

Here refinement corresponds to the *diminishing uncertainty* of agents as opposed to program refinement where detail is added to a specification. Still programme refinement is a more deterministic system which agrees with the notion of diminishing uncertainty.
Back to the example
Future event logic: the language $\mathcal{L}_{\rhd}$

**Syntax**

Given a finite set of agents $A$ and a set of propositional atoms $P$, the language of $\mathcal{L}_{\rhd}$ is inductively defined as

$$\varphi ::= p \mid \neg \varphi \mid (\varphi \land \varphi) \mid \Box a \varphi \mid \rhd a \varphi$$

where $a \in A$ and $p \in P$.

**Semantics**

$$M_s \models \rhd a \varphi \text{ iff for all } M'_s \preceq_a M_s, M'_s \models \varphi$$

Write $\rhd a \varphi$ for $\neg \rhd a \neg \varphi$. It is true now, iff there is an unspecified informative event for agent $a$, or $a$-refinement, after which $\varphi$ is true.
Future event logic: the language $\mathcal{L}^\mu_\triangleright$

**Syntax**

Given a finite set of agents $A$ and a set of propositional atoms $P$, the language of $\mathcal{L}_\triangleright$ is inductively defined as

$$\varphi ::= p | \neg \varphi | (\varphi \land \varphi) | \Box_a \varphi | \triangleright_a \varphi | \mu x. \varphi$$

where $a \in A$ and $p \in P$.

**Semantics**

$$M_s \models \triangleright_a \varphi \text{ iff for all } M_{s'} \leq_a M_s, \ M_{s'} \models \varphi$$

Write $\triangleright_a \varphi$ for $\neg \triangleright_a \neg \varphi$. It is true now, iff there is an unspecified informative event for agent $a$, or $a$-refinement, after which $\varphi$ is true.

Write $\nu x. \varphi$ for $\neg \mu x. \neg \varphi(\neg x)$
Example: Knowledge and belief

An informative event is possible after which agent $a$ knows that $p$ but agent $b$ does not know that.

$$\triangleright_a(\Box_a p \land \neg \Box_b \Box_a p)$$
Example:

Let $S$ be a discrete-event system with two possible actions $c$ and action $u$. Fix a formula $\varphi$ (say in the modal $\mu$-calculus).
Example: Open system – Module Checking

Let $S$ be a discrete-event system with two possible actions $c$ and action $u$. Fix a formula $\varphi$ (say in the modal $\mu$-calculus).

Interpret action $c$ as the moves of the system and action $u$ as the moves of an environment.

\[ S \models \blacktriangleleft u(\varphi) \]

iff

$S$ satisfies $\varphi$ in any environment.
Example: Open system – Module Checking

Let $S$ be a discrete-event system with two possible actions $c$ and action $u$. Fix a formula $\varphi$ (say in the modal $\mu$-calculus).

Interprete action $c$ as the moves of the system and action $u$ as the moves of an environment.

$$S \models \Box_u (\text{LiveEnv} \Rightarrow \varphi)$$

iff

$S$ satisfies $\varphi$ in any “live” environment

where $\text{LiveEnv} = \nu x. \Diamond_u \top \land \Box x$
Example: Basic Control Problems

Let $S$ be a discrete-event system with two possible actions $c$ and action $u$. Fix a formula $\varphi$ (say in the modal $\mu$-calculus).

Interprete action $c$ as controllable and action $u$ as uncontrollable.

$$S \models \Diamond_c(\varphi)$$
iff

there is a way to control $S$ to guarantee $\varphi$
Example: Controller Problems for Open Systems

Let $S$ be a discrete-event system with two possible actions $c$ and action $u$. Fix a formula $\varphi$ (say in the modal $\mu$-calculus).

Interprete action $c$ as controllable and action $u$ as uncontrollable.

$$\models S |\Rightarrow c \Rightarrow u \Rightarrow \text{LiveEnv} \Rightarrow \varphi$$

iff

there is a way to control the open system $S$ in any live environment so that the resulting satisfies $\varphi$.
And now?

- **FEL**: Axiomatization of $\mathcal{L}_\triangleright$
  - Soudness
  - Completeness

- **FEL\(_\mu\)**: Axiomatization of $\mathcal{L}^{\mu}_\triangleright$
  - Soudness
  - Completeness

- Complexity upper-bound for $\mathcal{L}_\triangleright$

- Succinctness of $\mathcal{L}_\triangleright$

- The logics $\mathcal{L}^C_\triangleright$ where $C$ is a fixed class of models ($S5$, $K4$, $\ldots$)
**FEL**: Axiomatization of $\mathcal{L}_\triangleright$ (one agent)

- **P**: All tautologies of propositional logic
- **K**: $\Box(\varphi \rightarrow \psi) \rightarrow \Box\varphi \rightarrow \Box\psi$
- **MP**: From $\vdash \varphi \rightarrow \psi$ and $\vdash \varphi$ infer $\vdash \psi$
- **Nec1**: From $\vdash \varphi$ infer $\vdash \Box\varphi$
FEL: Axiomatization of $\mathcal{L}_\triangleright$ (one agent)

- **P**  All tautologies of propositional logic
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- **Nec1** From $\vdash \varphi$ infer $\vdash \Box\varphi$
- **Nec2** From $\vdash \varphi$ infer $\vdash \triangleright\varphi$
**FEL: Axiomatization of** $\mathcal{L}_\triangleright$ **(one agent)**

P: All tautologies of propositional logic

K: $\Box(\varphi \rightarrow \psi) \rightarrow \Box \varphi \rightarrow \Box \psi$

MP: From $\vdash \varphi \rightarrow \psi$ and $\vdash \varphi$ infer $\vdash \psi$

Nec1: From $\vdash \varphi$ infer $\vdash \Box \varphi$

Nec2: From $\vdash \varphi$ infer $\vdash \triangleright \varphi$

G0: $\triangleright(\varphi \rightarrow \psi) \rightarrow \triangleright \varphi \rightarrow \triangleright \psi$
FEL: Axiomatization of $\mathcal{L}_\triangleright$ (one agent)

- **P**  All tautologies of propositional logic
- **K**  $\Box(\varphi \rightarrow \psi) \rightarrow \Box \varphi \rightarrow \Box \psi$
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- **Nec1**  From $\vdash \varphi$ infer $\vdash \Box \varphi$
- **Nec2**  From $\vdash \varphi$ infer $\vdash \Box \varphi$
- **G0**  $\Box(\varphi \rightarrow \psi) \rightarrow \Box \varphi \rightarrow \Box \psi$
- **G1**  $\alpha \leftrightarrow \Box \alpha$ where $\alpha$ is a propositional formula
**FEL: Axiomatization of** $\mathcal{L}_\triangleright$ **(one agent)**

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- **MP** From $\vdash \varphi \rightarrow \psi$ and $\vdash \varphi$ infer $\vdash \psi$
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**Nec2** From $\vdash \varphi$ infer $\vdash \triangleright \varphi$

- **G0** $\triangleright (\varphi \rightarrow \psi) \rightarrow \triangleright \varphi \rightarrow \triangleright \psi$
- **G1** $\alpha \leftrightarrow \triangleright \alpha$ where $\alpha$ is a propositional formula
- **GK** $\bigwedge_{\gamma \in \Gamma} \triangleright \gamma \leftrightarrow \triangleright (\Box \bigvee_{\gamma \in \Gamma} \gamma \wedge \bigwedge_{\gamma \in \Gamma} \hat{\Diamond} \gamma)$

\[ \triangleright \Gamma \text{ Cover operator} \]

(Where $\Gamma$ is a finite set of formulas)
Soundness of **FEL**

- **P**  All tautologies of propositional logic
- **K**  $\Box(\varphi \rightarrow \psi) \rightarrow \Box\varphi \rightarrow \Box\psi$
- **MP** From $\vdash \varphi \rightarrow \psi$ and $\vdash \varphi$ infer $\vdash \psi$
- **Nec1** From $\vdash \varphi$ infer $\vdash \Box\varphi$
- **Nec2** From $\vdash \varphi$ infer $\vdash \lozenge\varphi$
- **G0** $\lozenge(\varphi \rightarrow \psi) \rightarrow \lozenge\varphi \rightarrow \lozenge\psi$
- **G1** $\alpha \iff \lozenge\alpha$ where $\alpha$ is a propositional formula
- **GK** $\bigwedge_{\gamma \in \Gamma} \Diamond\lozenge\gamma \iff \lozenge\lozenge\Gamma$

**Theorem**

The axiomatization **FEL** is sound for $\mathcal{L}_\triangleright$.

**Proof sketch**

- **P**, **K**, **MP** and **Nec1** are sound (all models of $\mathcal{L}_\triangleright$ are models of $\mathcal{L}$)
Soundness of **FEL**

**Nec2**  From $\vdash \varphi$ infer $\vdash \varphi$

**G0**  $\varphi \rightarrow \psi \rightarrow \varphi \rightarrow \psi$

**G1**  $\alpha \leftrightarrow \varphi$ where $\alpha$ is a propositional formula

**GK**  $\bigwedge_{\gamma \in \Gamma} \varphi \leftrightarrow \varphi$

**Theorem**

The axiomatization **FEL** is sound for $\mathcal{L}_\mathcal{D}$.

**Proof sketch**

- **Nec2, G0 and G1** are easy
Soundness of FEL

\begin{itemize}
  \item **Nec2** From $\vdash \varphi$ infer $\vdash \Box \varphi$
  \item **G0** $\Box (\varphi \rightarrow \psi) \rightarrow \Box \varphi \rightarrow \Box \psi$
  \item **G1** $\alpha \leftrightarrow \Box \alpha$ where $\alpha$ is a propositional formula
  \item **GK** $\bigwedge_{\gamma \in \Gamma} \Diamond \Diamond \gamma \leftrightarrow \Diamond \nabla \Gamma$
\end{itemize}

**Theorem**

The axiomatization FEL is sound for $\mathcal{L}_\triangleright$.

**Proof sketch**

- **GK** not that hard
Completeness of FEL

Lemma

Every formula of $\mathcal{L}_\triangleright$ is logically equivalent to a formula of $\mathcal{L}$.

Proof

First, use cover logic instead of $\mathcal{L}$ ([Bilkova, M., A. Palmigiano and Y. Venema])

$$\phi ::= \bot | T | \phi \lor \phi | p \land \phi | \neg p \land \phi | \triangledown \Gamma$$

Indeed $\Box \phi$ iff $\triangledown \emptyset \lor \triangledown \{ \phi \}$, and $\Diamond \phi$ iff $\triangledown \{ \phi, T \}$. Axiom $\text{GK}$ now takes shape

$$\text{GK} \land \ \bigwedge_{\gamma \in \Gamma} \triangledown \{ \triangleright \gamma, T \} \leftrightarrow \triangleright \triangledown \Gamma$$
Completeness of FEL

Lemma
Every formula of $\mathcal{L}_\triangleright$ is logically equivalent to a formula of $\mathcal{L}$.

Proof
Second, show that given $\psi$ in cover logic with refinement, $\psi$ is equivalent to an $\triangleright$-free formula, and therefore to a formula in $\mathcal{L}$. We use equivalences:

- $\triangleright\bot$ iff $\bot$, $\triangleright\top$ iff $\top$, $\triangleright(p \land \varphi)$ iff $p \land \triangleright\varphi$ (refinements do not affect atoms), $\triangleright(\neg p \land \varphi)$ iff $\neg p \land \triangleright\varphi$.
- $\triangleright(\varphi \lor \psi)$ iff $\triangleright\varphi \lor \triangleright\psi$ (directly from the semantics of $\triangleright$).
- $\triangleright\nabla\Gamma$ iff $\bigwedge_{\gamma \in \Gamma} \nabla\{\triangleright\gamma, \top\}$ (GK)

Corollary
Let $\varphi \in \mathcal{L}_\triangleright$ be given and $\psi \in \mathcal{L}$ be equivalent to $\varphi$. If $\psi$ is a theorem in $K$, then $\varphi$ is a theorem in FEL.
**FEL**$^\mu$: Axiomatization of $L^\mu_\triangleright$ (one agent)

\[
\begin{align*}
P & \quad \text{All tautologies of propositional logic} \\
K & \quad \Box(\varphi \rightarrow \psi) \rightarrow \Box \varphi \rightarrow \Box \psi \\
MP & \quad \text{From } \vdash \varphi \rightarrow \psi \text{ and } \vdash \varphi \text{ infer } \vdash \psi \\
Nec1 & \quad \text{From } \vdash \varphi \text{ infer } \vdash \Box \varphi \\
Nec2 & \quad \text{From } \vdash \varphi \text{ infer } \vdash \triangleright \varphi \\
G0 & \quad \triangleright(\varphi \rightarrow \psi) \rightarrow \triangleright \varphi \rightarrow \triangleright \psi \\
G1 & \quad \alpha \leftrightarrow \triangleright \alpha \text{ where } \alpha \text{ is a propositional formula} \\
GK & \quad \land_{\gamma \in \Gamma} \lozenge \lozenge \gamma \leftrightarrow \lozenge \lozenge \Gamma
\end{align*}
\]
**FEL**$^\mu$: Axiomatization of $L^\mu_\triangleright$ (one agent)

- **P** All tautologies of propositional logic
- **K** $\Box(\varphi \rightarrow \psi) \rightarrow \Box \varphi \rightarrow \Box \psi$
- **MP** From $\vdash \varphi \rightarrow \psi$ and $\vdash \varphi$ infer $\vdash \psi$
- **Nec1** From $\vdash \varphi$ infer $\vdash \Box \varphi$
- **Nec2** From $\vdash \varphi$ infer $\vdash \Diamond \varphi$
- **G0** $\Diamond(\varphi \rightarrow \psi) \rightarrow \Diamond \varphi \rightarrow \Diamond \psi$
- **G1** $\alpha \leftrightarrow \Diamond \alpha$ where $\alpha$ is a propositional formula
- **GK** $\bigwedge_{\gamma \in \Gamma} \Diamond \Diamond \gamma \leftrightarrow \Diamond \Box \Diamond \Gamma$

Axiom and rule for the modal $\mu$-calculus:
- **F1** $\varphi[\mu x . \varphi \backslash x] \rightarrow \mu x . \varphi$
- **F2** From $\varphi[\psi \backslash x] \rightarrow \psi$ infer $\mu x \varphi \rightarrow \psi$
**FEL\(\mu\): Axiomatization of \(L^{\mu}_\Delta\) (one agent)**

\[
\begin{align*}
P & \quad \text{All tautologies of propositional logic} \\
K & \quad \Box(\varphi \rightarrow \psi) \rightarrow \Box\varphi \rightarrow \Box\psi \\
\text{MP} & \quad \text{From } \vdash \varphi \rightarrow \psi \text{ and } \vdash \varphi \text{ infer } \vdash \psi \\
\text{Nec1} & \quad \text{From } \vdash \varphi \text{ infer } \vdash \Box\varphi \\
\text{Nec2} & \quad \text{From } \vdash \varphi \text{ infer } \vdash \Diamond\varphi \\
\text{G0} & \quad \Diamond(\varphi \rightarrow \psi) \rightarrow \Diamond\varphi \rightarrow \Diamond\psi \\
\text{G1} & \quad \alpha \leftrightarrow \Diamond\alpha \text{ where } \alpha \text{ is a propositional formula} \\
\text{GK} & \quad \bigwedge_{\gamma \in \Gamma} \Diamond\Diamond\gamma \leftrightarrow \Diamond\Diamond\Diamond\Gamma
\end{align*}
\]

Axiom and rule for the modal \(\mu\)-calculus:

**F1** \(\varphi[\mu x. \varphi \setminus x] \rightarrow \mu x. \varphi\)

**F2** From \(\varphi[\psi \setminus x] \rightarrow \psi\) infer \(\mu x \varphi \rightarrow \psi\)

Two new interaction axioms:

**G3** \(\Diamond \mu x. \varphi \leftrightarrow \mu x. \Diamond\varphi\)
**FEL**: Axiomatization of $L_\mu^\triangleright$ (one agent)

**FEL**

\[
\begin{align*}
P & \quad \text{All tautologies of propositional logic} \\
K & \quad \Box(\varphi \rightarrow \psi) \rightarrow \Box\varphi \rightarrow \Box\psi \\
\text{MP} & \quad \text{From } \vdash \varphi \rightarrow \psi \text{ and } \vdash \varphi \text{ infer } \vdash \psi \\
\text{Nec1} & \quad \text{From } \vdash \varphi \text{ infer } \vdash \Box\varphi \\
\text{Nec2} & \quad \text{From } \vdash \varphi \text{ infer } \vdash \triangleright\varphi \\
\text{G0} & \quad \triangleright(\varphi \rightarrow \psi) \rightarrow \triangleright\varphi \rightarrow \triangleright\psi \\
\text{G1} & \quad \alpha \leftrightarrow \triangleright\alpha \text{ where } \alpha \text{ is a propositional formula} \\
\text{GK} & \quad \bigwedge_{\gamma \in \Gamma} \Diamond\triangleright\gamma \leftrightarrow \Diamond\nabla\Gamma
\end{align*}
\]

Axiom and rule for the modal $\mu$-calculus:

**F1** \ \ \ $\varphi[\mu x.\varphi/x] \rightarrow \mu x.\varphi$

**F2** \ \ \ From $\varphi[\psi/x] \rightarrow \psi$ infer $\mu x \varphi \rightarrow \psi$

Two new interaction axioms:

**G3** \ \ \ $\triangleright\mu x.\varphi \leftrightarrow \mu x.\triangleright\varphi$ where $\mu x.\varphi$ is a disjunctive formula
**FEL\(\mu\): Axiomatization of \(L\(\mu\)_\(\triangleright\) (one agent)**

\[
\begin{align*}
\text{P} & \quad \text{All tautologies of propositional logic} \\
\text{K} & \quad \Box(\varphi \rightarrow \psi) \rightarrow \Box\varphi \rightarrow \Box\psi \\
\text{MP} & \quad \text{From } \vdash \varphi \rightarrow \psi \text{ and } \vdash \varphi \text{ infer } \vdash \psi \\
\text{Nec1} & \quad \text{From } \vdash \varphi \text{ infer } \vdash \Box\varphi \\
\text{Nec2} & \quad \text{From } \vdash \varphi \text{ infer } \vdash \triangleright\varphi \\
\text{G0} & \quad \triangleright(\varphi \rightarrow \psi) \rightarrow \triangleright\varphi \rightarrow \triangleright\psi \\
\text{G1} & \quad \alpha \leftrightarrow \triangleright\alpha \text{ where } \alpha \text{ is a propositional formula} \\
\text{GK} & \quad \bigwedge_{\gamma \in \Gamma} \Diamond\Diamond\gamma \leftrightarrow \Diamond\triangleright\Gamma
\end{align*}
\]

Axiom and rule for the modal \(\mu\)-calculus:

- **F1** \(\varphi[\mu x.\varphi \backslash x] \rightarrow \mu x.\varphi\)
- **F2** From \(\varphi[\psi \backslash x] \rightarrow \psi\) infer \(\mu x\varphi \rightarrow \psi\)

Two new interaction axioms:

- **G3** \(\triangleright\mu x.\varphi \leftrightarrow \mu x.\triangleright\varphi\) where \(\mu x.\varphi\) is a **disjunctive formula**
- **G4** \(\triangleright\nu x.\varphi \leftrightarrow \nu x.\triangleright\varphi\) where \(\nu x.\varphi\) is a **df**
Disjunctive formulas

**Definition**

A *disjunctive formula (df)* is specified by the following abstract syntax:

$$\alpha ::= x \mid \alpha \lor \alpha \mid \mu x.\alpha \mid \nu x.\alpha \mid \pi \land \nabla \Gamma \mid \blacktriangleleft \alpha \mid \blacktriangleright \alpha$$

where $\pi$ is a conjunction of free literals (atoms or negated atoms, but not fixed-point variables)

$\mu$-disjunctive formulas are disjunctive formulas of $\mathcal{L}^\mu$ (the ones without $\blacktriangleleft$ or $\blacktriangleright$ operators)

**Proposition**

Every formula $\varphi$ of $\mathcal{L}^\mu$ is equivalent to a $\mu$-disjunctive formula
Soundness of $\text{FEL}^\mu$

$$\text{FEL} + \left\{ \begin{array}{ll}
\text{F1} & \varphi[\mu x.\varphi \setminus x] \rightarrow \mu x.\varphi \\
\text{F2} & \text{From } \varphi[\psi \setminus x] \rightarrow \psi \text{ infer } \mu x\varphi \rightarrow \psi \\
\text{G3} & \mathbf{\triangleright} \mu x.\varphi \leftrightarrow \mu x.\mathbf{\triangleright} \varphi \text{ where } \mu x.\varphi \text{ is a disjunctive formula} \\
\text{G4} & \mathbf{\triangleright} \nu x.\varphi \leftrightarrow \nu x.\mathbf{\triangleright} \varphi \text{ where } \nu x.\varphi \text{ is a df} \\
\end{array} \right. $$

The soundness of $\text{F1}$ and $\text{F2}$ are well known [Arnold, A. and D. Niwinski]

**Theorem**

The axioms $\text{G3}$ and $\text{G4}$ are sound.

**Proof sketch**

Very technical. Use bisimulation quantifiers [D’Agostino, G. and G. Lenzi, French, T.]
Soundness of $\text{FEL}^\mu$

$\text{FEL} + \begin{cases} 
\text{F1} & \varphi[\mu x.\varphi \setminus x] \rightarrow \mu x.\varphi \\
\text{F2} & \text{From } \varphi[\psi \setminus x] \rightarrow \psi \text{ infer } \mu x \varphi \rightarrow \psi \\
\text{G3} & \Box \mu x.\varphi \leftrightarrow \mu x.\Box \varphi \text{ where } \mu x.\varphi \text{ is a disjunctive formula} \\
\text{G4} & \Box \nu x.\varphi \leftrightarrow \nu x.\Box \varphi \text{ where } \nu x.\varphi \text{ is a df} 
\end{cases}$

The soundness of $\text{F1}$ and $\text{F2}$ are well known [Arnold, A. and D. Niwinski]

**Theorem**

The axioms $\text{G3}$ and $\text{G4}$ are sound.

**Proof sketch**

1. as $\mu z. f(z) = \bigcap \{z \mid f(z) \subseteq z\}$

   $\mu x.\varphi$ is equivalent to $\forall x (\Box (\varphi \rightarrow x) \rightarrow x)$

   “$\Box$” quantifies over all states in the model
Soundness of FEL\(^\mu\)

\[\text{FEL} + \left\{ \begin{array}{ll}
\text{F1} & \varphi[\mu x.\varphi \setminus x] \rightarrow \mu x.\varphi \\
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\text{G3} & \blacktriangle \mu x.\varphi \leftrightarrow \mu x.\blacktriangle \varphi \text{ where } \mu x.\varphi \text{ is a disjunctive formula} \\
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\end{array} \right.\]

The soundness of \textbf{F1} and \textbf{F2} are well known [Arnold, A. and D. Niwinski]

\begin{center}
\textbf{Theorem}
\end{center}

The axioms \textbf{G3} and \textbf{G4} are sound.

\begin{center}
\textbf{Proof sketch}
\end{center}

1. \(\mu x.\varphi\) is equivalent to \(\forall x(\blacksquare(\varphi \rightarrow x) \rightarrow x)\)
2. \(\nu x.\varphi\) is equivalent to \(\exists x(\blacksquare(x \rightarrow \varphi) \land x)\)

“\(\blacksquare\)” quantifies over all states in the model
Soundness of FEL\(^\mu\)

\[
\text{FEL} + \begin{cases} 
F1 & \varphi[\mu x.\varphi/x] \rightarrow \mu x.\varphi \\
F2 & \text{From } \varphi[\psi/x] \rightarrow \psi \text{ infer } \mu x\varphi \rightarrow \psi \\
G3 & \triangleright \mu x.\varphi \iff \mu x.\triangleright \varphi \text{ where } \mu x.\varphi \text{ is a disjunctive formula} \\
G4 & \triangleright \nu x.\varphi \iff \nu x.\triangleright \varphi \text{ where } \nu x.\varphi \text{ is a df}
\end{cases}
\]

The soundness of \(F1\) and \(F2\) are well known [Arnold, A. and D. Niwinski]

Theorem

The axioms \(G3\) and \(G4\) are sound.

Proof sketch

1. \(\mu x.\varphi\) is equivalent to \(\forall x(\Box(\varphi \rightarrow x) \rightarrow x)\)
2. \(\nu x.\varphi\) is equivalent to \(\exists x(\Box(x \rightarrow \varphi) \land x)\)
3. \(\triangleright \varphi\) is equivalent to \(\forall r \varphi^r\) and \(\triangleright \varphi\) is equivalent to \(\exists r \varphi^r\)

“\(\Box\)” quantifies over all states in the model

“\(\varphi^r\)” is the relativization of \(\varphi\) to the atom \(r\), which may be computed recursively by replacing every occurrence of \(\Box \psi\) in \(\varphi\) with \(\Box(r \rightarrow \psi^r)\)
Soundness of FEL\(^\mu\)

\[
\begin{align*}
\text{FEL} + \quad & \quad \text{F1} \quad \varphi[\mu x.\varphi] \rightarrow \mu x.\varphi \\
& \quad \text{F2} \quad \text{From } \varphi[\psi] \rightarrow \psi \text{ infer } \mu x\varphi \rightarrow \psi \\
& \quad \text{G3} \quad \Leftrightarrow \mu x.\varphi \leftrightarrow \mu x.\varnothing \varphi \text{ where } \mu x.\varphi \text{ is a disjunctive formula} \\
& \quad \text{G4} \quad \Leftrightarrow \nu x.\varphi \leftrightarrow \nu x.\varnothing \varphi \text{ where } \nu x.\varphi \text{ is a df}
\end{align*}
\]

The soundness of \textbf{F1} and \textbf{F2} are well known [Arnold, A. and D. Niwinski]

**Theorem**

The axioms \textbf{G3} and \textbf{G4} are sound.

**Proof sketch**

Example of \textbf{G3} (in contrapositive form \(\varnothing \nu x.\varphi \leftrightarrow \nu x.\varnothing \varphi\))

\[
\begin{align*}
\varnothing \nu x.\varphi & \iff \exists r \exists x(\Box(x \rightarrow \varphi) \land x)^r \\
& \iff \exists x \exists r(\Box(x \rightarrow \varphi^r) \land x) \\
& \iff \exists x(\exists r \Box(x \rightarrow \varphi^r) \land x) \\
& \iff \exists x(\Box \exists r(x \rightarrow \varphi^r) \land x) \quad \text{(you get } \iff \text{ only for disjunct. form.}) \\
& \iff \exists x(\Box(x \rightarrow \exists r \varphi^r) \land x) \\
& \iff \nu x.\varnothing \varphi
\end{align*}
\]
Completeness of $\text{FEL}^\mu$

**Lemma**

Every formula of $\mathcal{L}_\Delta^\mu$ is equivalent in $\text{FEL}_\mu$ to a formula of the modal $\mu$-calculus $\mathcal{L}^\mu$.

**Proof**

Remove the $\triangleright$ operators: use what we did for $\text{FEL}$ and

- $\triangleright \mu x. \varphi$ iff $\mu x. \triangleright \varphi$ (by G4 noting that all subformulas of a disjunctive formula are themselves disjunctive).
- $\triangleright \nu x. \varphi$ iff $\nu x. \triangleright \varphi$ (by G3).

**Corollary**

$\text{FEL}_\mu$ is complete for the logic $\mathcal{L}_\Delta^\mu$.
Complexity

From previous

- $L_\triangleright$ expressively equivalent to $L$ and $L_\triangleright^\mu$ expressively equivalent to $L^\mu$
- Decidability from computable translations, BUT disjunctive normal forms yield a non-elementary decision procedure

We have shown

- There is a 2EXPTIME upper-bound for $L_\triangleright$: we use a non-trivial tableau construction using a finite iteration of two-player games on some initial tableau
- $L_\triangleright$ and $L_\triangleright^\mu$ are exponentially more succinct than $L$ and $L^\mu$ respectively

Proposition

$L_\triangleright$ is able to express the property that two binary trees are $n$-bisimilar, with a formula of size $O(n^2)$.

(with $L^\mu$ you would need a formula of size exponential)
Perspectives

- Complexity:
  - For $\mathcal{L}_\triangleright$: what is the exact complexity?
Perspectives

Complexity:

- For $\mathcal{L}_{\triangleright}$: what is the exact complexity?
- For $\mathcal{L}^\mu_{\triangleright}$: a non-elementary upper-bound via bisimulation quantifiers. An elementary upper-bound by adapting the tableau construction for $\mathcal{L}_{\triangleright}$? Is there a 2EXPTIME lower-bound from $\mathcal{L}_{\triangleright}$?
Perspectives

- Complexity:
  - For $\mathcal{L}_{\rhd}$: what is the exact complexity?
  - For $\mathcal{L}^\mu_{\rhd}$: a non-elementary upper-bound via bisimulation quantifiers. An elementary upper-bound by adapting the tableau construction for $\mathcal{L}_{\rhd}$?
  - Is there a 2EXPTIME lower-bound from $\mathcal{L}_{\rhd}$?

- Varying classes of models on the axiomatization given:

  We note that while the schema FEL is sound for $\mathcal{L}_{\rhd}$, it is not the case that the axiom **GK**: $\bigwedge_{\gamma \in \Gamma} \lozenge \rhd \gamma \leftrightarrow \rhd \triangledown \Gamma$ is sound for restricted classes of models:
Perspectives

- **Complexity:**
  - For $\mathcal{L}_\triangleright$: what is the exact complexity?
  - For $\mathcal{L}_\triangleright^\mu$: a non-elementary upper-bound via bisimulation quantifiers. An elementary upper-bound by adapting the tableau construction for $\mathcal{L}_\triangleright$? Is there a 2EXPTIME lower-bound from $\mathcal{L}_\triangleright$?

- **Varying classes of models on the axiomatization given:**

  We note that while the schema $\textbf{FEL}$ is sound for $\mathcal{L}_\triangleright$, it is not the case that the axiom $\textbf{GK}$: $\bigwedge_{\gamma \in \Gamma} \boxdot\gamma \leftrightarrow \triangleright \gtrdot\Gamma$ is sound for restricted classes of models:

Consider the $\text{S}5$ frames

and $\Gamma = \{p \land \boxdot \neg p\}$:

\[\Diamond \triangleright \{p \land \Diamond \neg p\} \text{ is true (on the left) but not } \triangleright \Box (p \land \Diamond \neg p) \text{ (since } \Box (p \land \Diamond \neg p) \text{ is never true in reflexive frame)}\]
Perspectives

- Complexity:
  - For $\mathcal{L}_{\triangleright}$: what is the exact complexity?
  - For $\mathcal{L}^\mu_{\triangleright}$: a non-elementary upper-bound via bisimulation quantifiers. An elementary upper-bound by adapting the tableau construction for $\mathcal{L}_{\triangleright}$?
  - Is there a 2EXPTIME lower-bound from $\mathcal{L}_{\triangleright}$?

- Examine axiomatizations and complexity for refinement quantifiers in logics such as $\text{S5}$, $\text{KD45}$ and $\text{K4}$. 