Future event logic - axioms and complexity

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Informative Events

Consider a system that consists of a set of agents and a set of facts. The facts are known to be static so they do not change, although whether an agent *knows* a proposition is true may change.

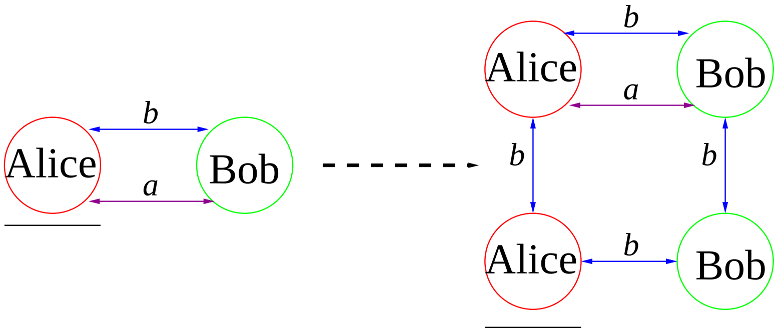
An agent may experience an informative event where their uncertainty in the system is reduced. An informative event is any change that updates a model in such a way that it is consistent with at least one of the "possibilities" inherent in the original model.

Examples of informative events include announcements (public or private), message passing systems and action models.

Example

Alice and Bob have both applied for a tenured lecturing position, and are waiting outside the Dean's office to hear which one has won the position. The Dean asks Alice to come into the office. He tells her she has won the position and she leaves.
This is an informative event. However, from Bob's point of view it has increased uncertainty. Previously he knew Alice did not know who had the job, but he considers it possible that she knows she has the job, that she knows she does not have the job, or that the Dean told her he has not yet made his decision.

Example



This graphic represents the effect of the informative event for Alice. Each circle represents a world where either Alice or Bob got the job, and an agent's uncertainty between which world is the actual world is represented by the relations. The underlined world is the actual world.

Technical preliminaries

A finite set of agents A and a countably infinite set of atoms P

Structures

$M = (S, R, V)$

- $S \ni s, t, \dots$ a *domain* of *states*
- $R : A \to \mathcal{P}(S \times S)$ *accessibility relation*; write $R_a(s, t)$
- $V : P \to \mathcal{P}(S)$ a *valuation*

For $s \in S$, M_s is a *state* or a *pointed Kripke model*.

Bisimulation

$M = (S, R, V)$ and $M' = (S', R', V')$. $\mathfrak{R} \subseteq S \times S'$ is a *bisimulation* whenever $(s, s') \in \mathfrak{R}$ if for all $a \in A$:

- **atoms** $s \in V(p)$ iff $s' \in V'(p)$ for all $p \in P$
- **forth-a** $\forall t \in sR_a, \exists t' \in s'R'_a$ with $(t, t') \in \mathfrak{R}$
- **back-a** vice versa

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Simulation

A relation that satisfies

- **atoms** $s \in V(p)$ iff $s' \in V'(p)$ for all $p \in P$
- **forth-a** for every $b \in A$

is a *simulation*. In that case $M'_{s'}$ is a *simulation* of M_s, and M_s is a *refinement* of $M'_{s'}$, and we write $M_s \preceq M'_{s'}$

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a-simulation ($a \in A$)

A relation that satisfies

- **atoms**
- **forth-a** for every $b \in A$
- **back-a** for every $b \in A - \{a\}$

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A relation that satisfies

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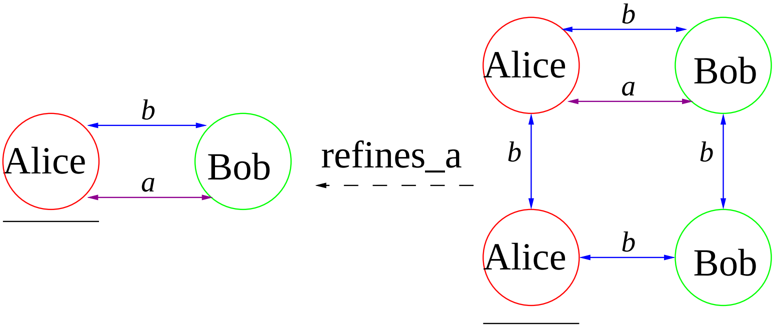
Here refinement corresponds to the *diminishing uncertainty* of agents as opposed to program refinement where detail is added to a specification. Still programme refinement is a more deterministic system which agrees with the notion of diminishing uncertainty.

Back to the example



Future event logic: the language $\mathcal{L}_{\triangleright}$

Syntax

Given a finite set of agents A and a set of propositional atoms P, the language of $\mathcal{L}_{\triangleright}$ is inductively defined as

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box_a \varphi \mid \blacktriangleright_a \varphi$$

where $a \in A$ and $p \in P$.

Semantics

$$M_s \models \blacktriangleright_a \varphi \text{ iff for all } M'_{s'} \preceq_a M_s,\ M'_{s'} \models \varphi$$

Write $\triangleright_a \varphi$ for $\neg \blacktriangleright_a \neg \varphi$. It is true now, iff there is an unspecified informative event for agent a, or a-refinement, after which φ is true.

Future event logic: the language $\mathcal{L}^{\mu}_{\triangleright}$

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Given a finite set of agents A and a set of propositional atoms P, the language of $\mathcal{L}_{\triangleright}$ is inductively defined as

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box_a\varphi \mid \blacktriangleright_a\varphi \mid \mu x.\varphi$$

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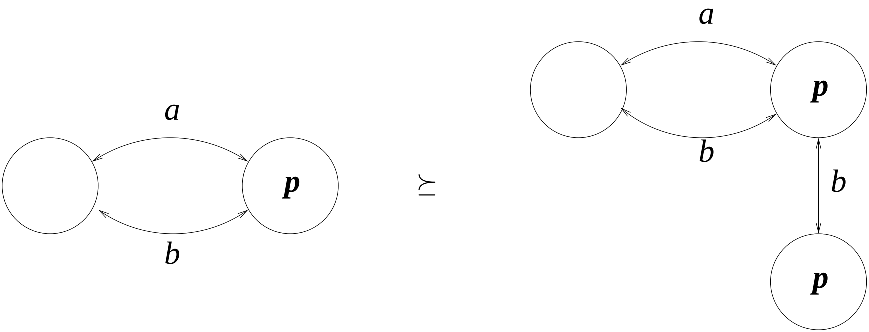
Write $\nu x.\varphi$ for $\neg\mu x.\neg\varphi(\neg x)$

Example: Knowledge and belief



An informative event is possible after which agent a knows that p but agent b does not know that.

$$\triangleright_a(\Box_a p \wedge \neg\Box_b\Box_a p)$$

Example:

Let S be a discrete-event system with two possible actions c and action u. Fix a formula φ (say in the modal μ-calculus).

Example: Open system – Module Checking

Let S be a discrete-event system with two possible actions c and action u. Fix a formula φ (say in the modal μ-calculus).

Interprete action c as the moves of the system and action u as the moves of an environment.

$$S \models \blacktriangleright_u(\varphi)$$

iff

S satisfies φ in any environment

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Let S be a discrete-event system with two possible actions c and action u. Fix a formula φ (say in the modal μ-calculus).

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$$S \models \blacktriangleright_u(\texttt{LiveEnv} \Rightarrow \varphi)$$

iff

S satisfies φ in any "live" environment

where $\texttt{LiveEnv} = \nu x.\Diamond_u \top \wedge \Box x$

Example: Basic Control Problems

Let S be a discrete-event system with two possible actions c and action u. Fix a formula φ (say in the modal μ-calculus).

Interprete action c as controllable and action u as uncontrollable.

$$S \models \triangleright_c(\varphi)$$

iff

there is a way to control S to guarantee φ

Example: Controller Problems for Open Systems

Let S be a discrete-event system with two possible actions c and action u. Fix a formula φ (say in the modal μ-calculus).

Interprete action c as controllable and action u as uncontrollable.

$$S \models \triangleright_c \blacktriangleright_u (\texttt{LiveEnv} \Rightarrow \varphi)$$

iff

there is a way to control the open system S

in any live environment

so that the resulting satisfies φ

And now?

- **FEL**: Axiomatization of $\mathcal{L}_{\triangleright}$
  - Soudness
  - Completeness
- **FEL**$_{\mu}$: Axiomatization of $\mathcal{L}^{\mu}{}_{\triangleright}$
  - Soudness
  - Completeness
- Complexity upper-bound for $\mathcal{L}_{\triangleright}$
- Succinctness of $\mathcal{L}_{\triangleright}$
- The logics $\mathcal{L}^{\mathcal{C}}_{\triangleright}$ where $\mathcal{C}$ is a fixed class of models ($S5$, $K4$, ...)

FEL: Axiomatization of $\mathcal{L}_{\triangleright}$ (one agent)

P	All tautologies of propositional logic
K	$\Box(\varphi \rightarrow \psi) \rightarrow \Box\varphi \rightarrow \Box\psi$
MP	From $\vdash \varphi \rightarrow \psi$ and $\vdash \varphi$ infer $\vdash \psi$
Nec1	From $\vdash \varphi$ infer $\vdash \Box\varphi$

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GK	$\bigwedge_{\gamma\in\Gamma} \Diamond\triangleright\gamma \leftrightarrow \triangleright\underbrace{(\Box \bigvee_{\gamma\in\Gamma} \gamma \wedge \bigwedge_{\gamma\in\Gamma} \Diamond\gamma)}_{\nabla\Gamma \text{ Cover operator}}$

(where Γ is a finite set of formulas)

Soundness of **FEL**

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K	$\Box(\varphi \to \psi) \to \Box\varphi \to \Box\psi$
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GK	$\bigwedge_{\gamma\in\Gamma} \Diamond\triangleright\gamma \leftrightarrow \triangleright\nabla\Gamma$

Theorem

The axiomatization **FEL** is sound for $\mathcal{L}_{\triangleright}$.

Proof sketch

- **P**, **K**, **MP** and **Nec1** are sound (all models of $\mathcal{L}_{\triangleright}$ are models of $\mathcal{L}$)

Soundness of FEL

Nec2 From $\vdash \varphi$ infer $\vdash \blacktriangleright\varphi$
G0 $\blacktriangleright(\varphi \to \psi) \to \blacktriangleright\varphi \to \blacktriangleright\psi$
G1 $\alpha \leftrightarrow \blacktriangleright\alpha$ where α is a propositional formula
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Theorem

The axiomatization **FEL** is sound for $\mathcal{L}_{\triangleright}$.

Proof sketch

- **Nec2**, **G0** and **G1** are easy

Soundness of **FEL**

Nec2	From $\vdash \varphi$ infer $\vdash \blacktriangleright\varphi$
G0	$\blacktriangleright(\varphi \rightarrow \psi) \rightarrow \blacktriangleright\varphi \rightarrow \blacktriangleright\psi$
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The axiomatization **FEL** is sound for $\mathcal{L}_{\triangleright}$.

Proof sketch

- **GK** not that hard

Completeness of **FEL**

Lemma

Every formula of $\mathcal{L}_{\triangleright}$ is logically equivalent to a formula of $\mathcal{L}$.

Proof

First, use cover logic instead of $\mathcal{L}$ ([Bilkova, M., A. Palmigiano and Y. Venema])

$$\varphi \ ::= \ \bot \mid \top \mid \varphi \vee \varphi \mid p \wedge \varphi \mid \neg p \wedge \varphi \mid \nabla\Gamma$$

Indeed $\Box\varphi$ iff $\nabla\emptyset \vee \nabla\{\varphi\}$, and $\Diamond\varphi$ iff $\nabla\{\varphi, \top\}$. Axiom **GK** now takes shape

$$\textbf{GK} \quad \bigwedge_{\gamma\in\Gamma} \nabla\{\triangleright\gamma, \top\} \leftrightarrow \triangleright\nabla\Gamma$$

Completeness of **FEL**

Lemma

Every formula of $\mathcal{L}_\triangleright$ is logically equivalent to a formula of $\mathcal{L}$.

Proof

Second, show that given ψ in cover logic with refinement, ψ is equivalent to an $\triangleright$-free formula, and therefore to a formula in $\mathcal{L}$. We use equivalences:

- $\triangleright\bot$ iff $\bot$, $\triangleright\top$ iff $\top$, $\triangleright(p \wedge \varphi)$ iff $p \wedge \triangleright\varphi$ (refinements do not affect atoms), $\triangleright(\neg p \wedge \varphi)$ iff $\neg p \wedge \triangleright\varphi$.
- $\triangleright(\varphi \vee \psi)$ iff $\triangleright\varphi \vee \triangleright\psi$ (directly from the semantics of $\triangleright$).
- $\triangleright\nabla\Gamma$ iff $\bigwedge_{\gamma\in\Gamma} \nabla\{\triangleright\gamma, \top\}$ (**GK**)

Corollary

Let $\varphi \in \mathcal{L}_\triangleright$ be given and $\psi \in \mathcal{L}$ be equivalent to φ. If ψ is a theorem in **K**, then φ is a theorem in **FEL**.

FEL$^{\mu}$: Axiomatization of $\mathcal{L}^{\mu}{}_{\triangleright}$ (one agent)

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Axiom and rule for the modal μ-calculus:

F1 $\varphi[\mu x.\varphi \backslash x] \to \mu x.\varphi$

F2 From $\varphi[\psi \backslash x] \to \psi$ infer $\mu x \varphi \to \psi$

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Two new interaction axioms:

G3 $\blacktriangleright \mu x.\varphi \leftrightarrow \mu x.\blacktriangleright\varphi$

FEL$^\mu$: Axiomatization of $\mathcal{L}^{\mu}{}_{\triangleright}$ (one agent)

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Two new interaction axioms:

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G4 $\blacktriangleright\nu x.\varphi \leftrightarrow \nu x.\blacktriangleright\varphi$ where $\nu x.\varphi$ is a *df*

Disjunctive formulas

Definition

A *disjunctive formula (df)* is specified by the following abstract syntax:

$$\alpha ::= x \mid \alpha \vee \alpha \mid \mu x.\alpha \mid \nu x.\alpha \mid \pi \wedge \nabla\Gamma \mid \blacktriangleright\alpha \mid \triangleright\alpha$$

where π is a conjunction of free literals (atoms or negated atoms, but not fixed-point variables)

μ-disjunctive formulas are disjunctive formulas of $\mathcal{L}^{\mu}$ (the ones without $\blacktriangleright$ or $\triangleright$ operators)

Proposition

Every formula φ of $\mathcal{L}^{\mu}$ is equivalent to a μ-disjunctive formula

Soundness of FEL$^{\mu}$

$$\textbf{FEL} + \begin{cases} \textbf{F1} & \varphi[\mu x.\varphi \backslash x] \rightarrow \mu x.\varphi \\ \textbf{F2} & \text{From } \varphi[\psi \backslash x] \rightarrow \psi \text{ infer } \mu x \varphi \rightarrow \psi \\ \textbf{G3} & \blacktriangleright \mu x.\varphi \leftrightarrow \mu x.\blacktriangleright \varphi \text{ where } \mu x.\varphi \text{ is a } \textit{disjunctive formula} \\ \textbf{G4} & \blacktriangleright \nu x.\varphi \leftrightarrow \nu x.\blacktriangleright \varphi \text{ where } \nu x.\varphi \text{ is a } \textit{df} \end{cases}$$

The soundness of **F1** and **F2** are well known [Arnold, A. and D. Niwinski]

Theorem

The axioms **G3** and **G4** are sound.

Proof sketch

Very technical. Use bisimulation quantifiers [D'Agostino, G. and G. Lenzi, French, T.]

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1. as $\mu z.f(z) = \bigcap\{z \mid f(z) \subseteq z\}$
   $\mu x.\varphi$ is equivalent to $\forall x(\blacksquare(\varphi \rightarrow x) \rightarrow x)$

"$\blacksquare$" quantifies over all states in the model

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1. $\mu x.\varphi$ is equivalent to $\forall x(\blacksquare(\varphi \to x) \to x)$
2. $\nu x.\varphi$ is equivalent to $\exists x(\blacksquare(x \to \varphi) \wedge x)$

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1. $\mu x.\varphi$ is equivalent to $\forall x(\blacksquare(\varphi \rightarrow x) \rightarrow x)$
2. $\nu x.\varphi$ is equivalent to $\exists x(\blacksquare(x \rightarrow \varphi) \wedge x)$
3. $\blacktriangleright\varphi$ is equivalent to $\forall r \varphi^r$ and $\triangleright\varphi$ is equivalent to $\exists r \varphi^r$

"$\blacksquare$" quantifies over all states in the model
"φ^r" is the relativization of φ to the atom r, which may be computed recursively by replacing every occurrence of $\Box\psi$ in φ with $\Box(r \rightarrow \psi^r)$

Soundness of FEL$^{\mu}$

FEL $+$
- **F1** $\varphi[\mu x.\varphi \backslash x] \to \mu x.\varphi$
- **F2** From $\varphi[\psi \backslash x] \to \psi$ infer $\mu x \varphi \to \psi$
- **G3** $\blacktriangleright \mu x.\varphi \leftrightarrow \mu x.\blacktriangleright \varphi$ where $\mu x.\varphi$ is a *disjunctive formula*
- **G4** $\blacktriangleright \nu x.\varphi \leftrightarrow \nu x.\blacktriangleright \varphi$ where $\nu x.\varphi$ is a *df*

The soundness of **F1** and **F2** are well known [Arnold, A. and D. Niwinski]

Theorem

The axioms **G3** and **G4** are sound.

Proof sketch

Example of **G3** (in contrapositive form $\triangleright \nu x.\varphi \leftrightarrow \nu x.\triangleright \varphi$)

$$
\begin{array}{lll}
\triangleright \nu x.\varphi & \leftrightarrow & \exists r \exists x(\blacksquare(x \to \varphi) \wedge x)^r \\
 & \leftrightarrow & \exists x \exists r(\blacksquare(x \to \varphi^r) \wedge x) \\
 & \leftrightarrow & \exists x(\exists r \blacksquare(x \to \varphi^r) \wedge x) \\
 & \to & \exists x(\blacksquare \exists r(x \to \varphi^r) \wedge x) \quad \text{(you get } \leftarrow \text{ only for disjunct. form.)} \\
 & \leftrightarrow & \exists x(\blacksquare(x \to \exists r \varphi^r) \wedge x) \\
 & \leftrightarrow & \nu x.\triangleright \varphi
\end{array}
$$

Completeness of FEL$^\mu$

Lemma

Every formula of $\mathcal{L}^\mu_\triangleright$ is equivalent in $\mathbf{FEL}_\mu$ to a formula of the modal μ-calculus $\mathcal{L}^\mu$.

Proof

Remove the $\triangleright$ operators: use what we did for **FEL** and

- $\triangleright\mu x.\varphi$ iff $\mu x.\triangleright\varphi$ (by **G4** noting that all subformulas of a disjunctive formula are themselves disjunctive).
- $\triangleright\nu x.\varphi$ iff $\nu x.\triangleright\varphi$ (by **G3**).

Corollary

$\mathbf{FEL}_\mu$ is complete for the logic $\mathcal{L}^\mu_\triangleright$

Complexity

From previous

- $\mathcal{L}_{\triangleright}$ expressively equivalent to $\mathcal{L}$ and $\mathcal{L}^{\mu}_{\triangleright}$ expressively equivalent to $\mathcal{L}^{\mu}$
- Decidability from computable translations, BUT disjunctive normal forms yield a non-elementary decision procedure

We have shown

- There is a 2EXPTIME upper-bound for $\mathcal{L}_{\triangleright}$: we use a non-trivial tableau construction using a finite iteration of two-player games on some initial tableau
- $\mathcal{L}_{\triangleright}$ and $\mathcal{L}^{\mu}_{\triangleright}$ are exponentially more succinct than $\mathcal{L}$ and $\mathcal{L}^{\mu}$ respectively

Proposition

$\mathcal{L}_{\triangleright}$ is able to express the property that two binary trees are n-bisimilar, with a formula of size $O(n^2)$.

(with $\mathcal{L}^{\mu}$ you would need a formula of size exponential)

Perspectives

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  - For $\mathcal{L}_{\triangleright}$: what is the exact complexity?

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- Varying classes of models on the axiomatization given:

  We note that while the schema **FEL** is sound for $\mathcal{L}_{\triangleright}$, it is not the case that the axiom **GK:** $\bigwedge_{\gamma \in \Gamma} \Diamond \triangleright \gamma \leftrightarrow \triangleright \nabla \Gamma$ is sound for restricted classes of models:

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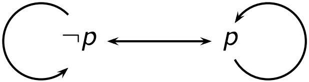
Consider the $S5$ frames



and $\Gamma = \{p \wedge \Diamond\neg p\}$:

$\Diamond\triangleright\{p \wedge \Diamond\neg p\}$ is true (on the left) but not $\triangleright\Box(p \wedge \Diamond\neg p)$ (since $\Box(p \wedge \Diamond\neg p)$ is never true in reflexive frame)

Perspectives

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  - For $\mathcal{L}_{\triangleright}$: what is the exact complexity?
  - For $\mathcal{L}^{\mu}_{\triangleright}$: a non-elementary upper-bound via bisimuation quantifiers. An elementary upper-bound by adapting the tableau construction for $\mathcal{L}_{\triangleright}$? Is there a 2EXPTIME lower-bound from $\mathcal{L}_{\triangleright}$?
- Examine axiomatizations and complexity for refinement quantifiers in logics such as **S5**, **KD45** and **K4**.