Goldblatt-Thomason-style Theorems for Graded Modal Language

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If you read a classic well in your own context, then you can obtain something new.

A classic of this talk:
- Fine, K.
  ‘In so many possible worlds’,

Our context: Semantical Characterization of Frame Definablity of GML

Something new: GbTh Theorem for GML
1. Introduction
   - GbTh Theorem for Basic ML
   - Kripke Semantics for GML

2. Neighborhood Semantical View to GML
   - \(g\)-bounded morphic images
   - Relative GbTh Theorem for GML

3. Graph Semantical View to GML
   - Graded ultrafilter images
   - GbTh Theorem for GML
GbTh Theorem for Basic ML

Let $F$ be a first-order definable class of frames. TFAE:

(A) $F$ is modally definable.

(B) $F$ is closed under taking

(i) generated subframes,

(ii) disjoint unions, and

(iii) bounded morphic images,

and $\overline{F}$ is closed under taking

(iv) ultrafilter extentions.
GbTh Theorem for GML

Let $F$ be a first-order definable class of frames. TFAE:

(A) $F$ is GML-definable.

(B) $F$ is closed under taking

(i) generated subframes,
(ii) disjoint unions,
(iii) $g$-bounded morphic images, and
(iv) graded ultrafilter images.
Propositional language extended with \( \{ \Diamond_k \mid k \in \mathbb{N} \} \).

Given any Kripke model \( (W, R, V) \),

\[
w \in \llbracket \Diamond_k \varphi \rrbracket \iff \#(R(w) \cap \llbracket \varphi \rrbracket) \geq k,
\]

where \( R(w) = \{ v \in W \mid wRv \} \).

\( \Diamond \varphi \equiv \Diamond_1 \varphi \).

\( \Diamond_k p \rightarrow \Diamond_k (p \lor q) \) is valid.

So, \( (\Diamond_k p \lor \Diamond_k q) \rightarrow \Diamond_k (p \lor q) \) is valid.

\( \Diamond_2 \top \) defines the existence of at least two successors for each state, which is undefinable in BML.
Non-Normal Character of $\diamond_k (k > 1)$
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Non-Normal Character of $\diamond_k \ (k > 1)$

- $\diamond_k (p \lor q) \rightarrow (\diamond_k p \lor \diamond_k q)$ is invalid ($k > 1$).
- $\diamond_k (p \lor q)$ is not equivalent to $\diamond_k p \lor \diamond_k q \ (k > 1)$. 
Derived Neighborhood Structure from Kripke Frame

\[ w \in \left[ \lozenge_{k} \varphi \right] \iff \#(R(w) \cap \left[ \varphi \right]) \geq k \]
\[ \iff \exists X \subseteq R(w). (\#X = k \text{ and } X \subseteq \left[ \varphi \right]) \]
\[ \iff \left[ \varphi \right] \in \tau_{k}(w), \]

where \( \tau_{k}(w) \) is defined as:

\[ Y \in \tau_{k}(w) \iff \exists X \subseteq R(w). (\#X = k \text{ and } X \subseteq Y). \]

Clearly, \( \tau_{k}(w) \) is monotone. For neighborhood structures, we have an appropriate notion of morphism.
**g-Bounded Morphism**

- Consider $(W, R)$ and $(W', R')$.
- $f : W \to W'$ is a *g*-bounded morphism if:

  $$f^{-1}[Y] \in \tau_k(w) \iff Y \in \tau'_k(f(w)) \quad (k \in \mathbb{N}),$$

  for any $w \in W$ and any $Y \subseteq W'$.
- $\mathcal{F}'$ is a *g*-bounded morphic image of $\mathcal{F}$ if there is a surjective *g*-bounded morphism from $\mathcal{F}$ to $\mathcal{F}'$.

If $\mathcal{F}'$ is a *g*-bounded morphic image of $\mathcal{F}$, then $\mathcal{F} \Vdash \varphi$ implies $\mathcal{F}' \Vdash \varphi$ for any $\varphi$ of GML.
If \( f \) is a \( g \)-bounded morphism from \( \mathcal{F} \) to \( \mathcal{F}' \) \iff \( f \) is a bounded morphism and \( f \upharpoonright R(w) \) is injective (\( w \in W \)).
Undefinability of Irreflexivity in GML

If $f$ is a $g$-bounded morphism from $\mathcal{F}$ to $\mathcal{F}'$ $\iff$ $f$ is a bounded morphism and $f \upharpoonright R(w)$ is injective ($w \in W$).
Graded Jankov-Fine Formula

- Let $\mathcal{F}$ be a finite transitive point-generated frame with the root $w$.
- Put $W = \{w_0, \ldots, w_n\}$ and $w_0 := w$.
- Associate each $w_i$ with a new propositional letter $p_i$ and define $p_X := \bigvee\{p_i \mid w_i \in X\}$ for $X \subseteq W$.
- The graded Jankov-Fine formula $\varphi_{\mathcal{F},w}$ is the conjunction of all the following:
  - $p_0$
  - $\square(p_0 \lor \cdots \lor p_n)$
  - $\land\{\square^+(p_i \rightarrow \neg p_j) \mid i \neq j\}$
  - $\land\{\square^+(p_i \rightarrow \Diamond_k p_X) \mid X \in \tau_k(w_i)\}$
  - $\land\{\square^+(p_i \rightarrow \neg \Diamond_k p_X) \mid X \notin \tau_k(w_i)\}$

where $\square^+ \psi = \psi \land \square \psi$. 
Relative GbTh Theorem for GML

(Lemma) Let \( G \) be a finite transitive point-generated frame with the root \( w \). Then, for any transitive \( G = (G, S) \), TFAE:

(A) \( G \not\models \neg \varphi_{G, w} \)

(B) \( \exists v \in G. G \) is a \( g \)-bounded morphic images of \( G_v \).

Let \( C \) be the class of all finite transitive frames and \( F \subseteq C \). TFAE:

(A) \( F \) is GML-definable within \( C \)

(B) \( F \) is closed under taking:

- generated subframes,
- (finite) disjoint unions,
- \( g \)-bounded morphic images.
Guiding Idea for Alternative of Ultrafilter Extension

In BML:

\[ \mathcal{F} \equiv \text{ue} \mathcal{F} \]

\[ \sqcap \leftrightarrow \sqcup \]

\[ \mathcal{F} \cong \mathcal{F} \]
Our Guiding Idea (cf. (S. & Sato 2007))

Completeness proof of extended modal logic gives us an idea of alternative of ultrafilter extensions.
Kit Fine’s completeness proof of GML (1972) gives us:

\[ \mathcal{F} \xleftarrow{\text{GRADED ULTRAFILTER IMAGES}} (\text{Uf}(W), (R_{k}^{ue})_{k \in \mathbb{N}}) \xleftarrow{\text{GbTh Theorem for GML}} \mathcal{G} \]

What are Graph Frame and Fine Mapping?

\[ \mathcal{F} \xleftarrow{\text{GRADED ULTRAFILTER IMAGES}} (\text{Uf}(W), (R_{k}^{ue})_{k \in \mathbb{N}}) \xleftarrow{\text{GbTh Theorem for GML}} \mathcal{G} \]
Graph Semantics for GML

$(W, (R_k)_{k \in \mathbb{N}})$ is a graph frame if $W \neq \emptyset$ and $R_k \subseteq W^2$ such that: $k < l$ implies $R_l \subseteq R_k$. 
(\(W, (R_k)_{k \in \mathbb{N}}\)) is a graph frame if \(W \neq \emptyset\) and \(R_k \subseteq W^2\) such that: \(k < l\) implies \(R_l \subseteq R_k\).
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![Diagram of graph frame](image)
Graph Semantics for GML

(W, (R_k)_{k \in \mathbb{N}}) is a graph frame if W \neq \emptyset and R_k \subseteq W^2 such that: k < l implies R_l \subseteq R_k.
Graph Semantics and Coalgebraic Semantics for GML

- Given any graph frame and any valuation \( V \), we define:

\[
\mathcal{G} \models_{V} \Diamond_{k} \varphi \iff \exists X \subseteq \omega \, |\varphi| \cdot \exists l : X \to \mathbb{N}. \left( \forall x \in X. [wR_{l(x)}x] \text{ and } \sum_{x \in X} l(x) \geq k \right),
\]

where \( |\varphi| := \{ w \in \mathcal{W} \mid w \models_{V} \varphi \} \).

- We can construct an equivalent coalgebraic model from a graph model, and vice versa, where the intended functor is the infinite multiset functor \( B_{\infty} = (\mathbb{N} \cup \{ \infty \})^{X} \).
Fine Mapping

1. Let \((W, (R_k)_{k \in \mathbb{N}})\) be a graph frame and \((G, S)\) a Kripke frame.
2. We say that \(f : G \rightarrow W\) is a Fine mapping if:
   \[
   \#(f^{-1}(w) \cap S(x)) \geq k \iff f(x) R_k w \quad (k \in \mathbb{N}),
   \]
   for any \(x \in G\) and any \(w \in W\).

If \(f : G \rightarrow W\) is a surjective Fine mapping,
then \((G, S) \models \varphi\) implies \((W, (R_k)_{k \in \mathbb{N}}) \models \varphi\)
for any \(\varphi\) of GML.
An Example of Fine Mapping: Fine (1972)

- Given any graph frame \((W, (R_k)_{k \in \mathbb{N}})\), define
  - \(G := W \times \mathbb{N}\).
  - \((w, k) R (w', l) \iff wR_l w'\).

- Then, \(\pi_1 : W \times \mathbb{N} \to W\) is a Fine mapping.

- Consider the following graph frame:
  - \(W = \{ \ast \}\)
  - \(R_0 = \{(\ast, \ast)\}\) and \(R_k = \emptyset \) \((k > 1)\).

- The above construction gives us:

- $G := W \times \mathbb{N}$.
- $(w, k)R(w', l) \iff wR_l w'$. 

![Diagram showing a directed graph with nodes labeled 0 to 7, illustrating the relation $(w, k)R(w', l) \iff wR_l w'$.]
Graded Ultrafilter Image

Given \((W, R)\), we define:

- \(\text{Uf}(W) := \text{the set of all ultrafilters on } W\).
- \(U_1 \sim_k U_2 \iff \forall X \subseteq W. [X \in U_2 \implies m_k(X) \in U_1]\),
  where \(m_k(X) = \{ w \in W \mid \#(R(w) \cap X) \geq k \}\).
- Then, \((\text{Uf}(W), (R^\text{ue}_k)_{k \in \mathbb{N}})\) is a graph frame.

\[(\text{Uf}(W), (R^\text{ue}_k)_{k \in \mathbb{N}}) \models \varphi \implies (W, R) \models \varphi \text{ for any } \varphi \text{ of GML.}\]

- \(\mathcal{F} = (W, R)\) is a graded ultrafilter image of \(\mathcal{G} = (G, S)\) if there exists a surjective Fine mapping \(f : G \to \text{Uf}(W)\):

\[
\mathcal{F} \ (\text{Uf}(W), (R^\text{ue}_k)_{k \in \mathbb{N}}) \leftarrow \mathcal{G}
\]
Undefinablity of $\forall x.\exists y. (xRy \text{ and } yRy)$
GbTh Theorem for GML

Let $F$ be a first-order definable class of frames. TFAE:

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(∵) Use the model-theoretic proof by Van Benthem (1993).
Further Directions

- GbTh Theorem for Graded **Hybrid** Language?
  - GbTh Theorem for **HL** by Ten Cate (2005).

- The Scope of Our Guiding Idea
  - Is it possible to get GbTh Theorem for conditional logic over preference frames? What else?

- The second author recently established that there is a natural GbTh-Theorem for coalgebraic semantics of GML (by the infinite multiset functor) via duality between algebras and coalgebras (cf. Kurz and Rosický (2007)).
Take-home Message

We revive Fine’s old idea in the new context of GbTh-style characterization.

Thank You